electronic-Liquid Crystal Communications February 10, 2007

Eigenvalue decomposition for tensors of arbitrary rank

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Abstract

In many physical situations, the eigenvalues and eigenvectors of tensors are of key importance. Methods for determining eigenvalues and eigenvectors and for implementing eigenvalue decomposition are well known for tensors of second rank. There are many physical situations, however, where knowledge of the eigenvalues and eigenvectors of tensors of higher rank tensors would be useful. We propose a procedure here for determining the eigenvalues and eigenvectors and for implementing eigenvalue decomposition of tensors of arbitrary rank.

I. INTRODUCTION

Physical quantities are tensors. Multiway arrays, closely related to tensors [1] are also of importance in image and signal processing [2]. In many physical situations, it is useful to express tensors in terms of their eigenvalues and eigenvectors. For example, in crystal optics, the refractive indices of optical eigenmodes are equal to the square roots of the eigenvalues of the dielectric tensor, and the associated eigenvectors define the polarization states [3]. In liquid crystal physics, the free energy of spatially homogeneous nematics is a function of the eigenvalues of the orientational order parameter tensor [4]; if the system is not homogeneous, the eigenvectors and their derivatives also appear. For nematic liquid crystals consisting of rod-like molecules, the relevant order parameter is $\langle l_\alpha l_\beta \rangle$, a second rank tensor, where l_α is a component of a unit vector along a rod and $\langle \cdot \rangle$ denotes the ensemble average. The three eigenvalues of the tensor characterize the three phases: isotropic, and uniaxial and biaxial nematic.

The eigenvalues are independent scalars which are invariant under rotations of the coordinate system. The free energy of a system with a tensor order parameter can be conveniently represented as function of the eigenvalues of the tensor. The eigenvalues may be regarded as scalar order parameters, giving a measure of different types of order. The eigenvalues of tensors thus play a crucial role in statistical descriptions of condensed matter systems.

Methods for determining eigenvalues and eigenvectors and for implementing eigenvalue decomposition are well known for second rank tensors [5]. There are many physical situations, however, where knowledge of eigenvalues and eigenvectors and eigenvalue decomposition of higher rank tensors would be useful. Examples are elasticity, where the elastic modulus is a fourth rank tensor, and liquid crystals, where the orientational order parameter of particles with higher symmetry have higher rank order parameters. For example, the order parameter for tetrahedral particles is a third rank tensor [6],[7],[8],[9], which may be written as $\langle l_\alpha l_\beta l_\gamma \rangle$. Although tensor decomposition is an active area of research today [1],[10], to our knowledge, a direct generalization of the process of finding eigenvalues and eigenvectors and of eigenvalue decomposition for tensors of rank greater than two does not exist.

We propose such a generalization here, and provide a procedure for determining the eigenvalues and eigenvectors, implementing eigenvalue decomposition of tensors of arbitrary rank.

II. BACKGROUND

Our proposed approach parallels the standard solution of the eigenvalue problem and standard eigenvalue decomposition tensors. We use the following notation. The rank r of a tensor refers to the number of its indices [11]. The dimension D refers to the number of values each index is allowed to take; in our notation, the indices take on values $1, 2, ..., D$. We take the dimension to be the same for all indices. We use the Einstein summation convention, where summation is implied over repeated Greek indices. For a second rank tensor $A_{\alpha\beta}$ which can be represented as a matrix, we use the convention that the first index refers to the row, and the second to the column.

The standard eigenvalue problem, often written as

$$\mathbf{Ax} = \lambda\mathbf{x} \tag{1}$$

is, in indicial notation,

$$A_{\alpha\beta}x^r_\beta = \lambda x^r_\alpha. \tag{2}$$

Here $A_{\alpha\beta}$ is a given second rank tensor, the scalar λ is an eigenvalue and the vector x^r_α is a right eigenvector, both to be determined. The standard solution is obtained [12] by writing Eq. (2) as

$$(A_{\alpha\beta} - \lambda\delta_{\alpha\beta})x^r_\beta = 0, \tag{3}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta, and noting that for a nontrivial solution to exist, the determinant must vanish. That is,

$$|A_{\alpha\beta} - \lambda\delta_{\alpha\beta}| = 0. \tag{4}$$

This gives the secular equation, a polynomial of order D in λ, set equal to zero, whose roots are the eigenvalues λ_i. The number of eigenvalues is equal to the dimension D. Once the eigenvalues are known, for each eigenvalue λ_i, Eq. (2) may be solved for the corresponding right eigenvector $x^r_{i\alpha}$.

One can also write the same eigenvalue problem in terms of the left eigenvector; that is

$$x^l_\alpha A_{\alpha\beta} = \lambda x^l_\beta. \tag{5}$$

We note here that the order of writing the symbols x^l_α and $A_{\alpha\beta}$ is immaterial; the superscripts l and r refer to the index of $A_{\alpha\beta}$ over which summation takes place. Proceeding as before, the secular equation is again

$$|A_{\alpha\beta} - \lambda\delta_{\alpha\beta}| = 0, \tag{6}$$

and so the set of eigenvalues for left and right eigenvectors are the same. By multiplying both sides of Eq. (2) by x^l_α, we see that $x^r_{i\alpha}x^l_{j\alpha} = 0$ if $i \neq j$, that is, left and right eigenvectors belonging to different eigenvalues are orthogonal. Since the magnitudes of the eigenvectors are undetermined, they may be conveniently normalized as shown below:

$$\hat{x}^r_{i\alpha} = \frac{x^r_{i\alpha}}{\sqrt{x^r_{i\nu}x^l_{i\nu}}} \tag{7}$$

and

$$\hat{x}^l_{i\alpha} = \frac{x^l_{i\alpha}}{\sqrt{x^r_{i\nu}x^l_{i\nu}}}. \tag{8}$$

It follows that

$$\hat{x}^l_{i\alpha}\hat{x}^r_{i\alpha} = 1. \tag{9}$$

If the eigenvalues are all different, $\mathbf{A}$ can be diagonalized. It is useful to define $\mathbf{X}^r$, the tensor of normalized right eigenvectors, with elements $\hat{x}^r_{i\alpha}$. $\mathbf{X}^r$ is defined as

$$X^r_{\alpha\beta} = \hat{x}^r_{i\alpha}, \tag{10}$$

where

$$\beta = i, \tag{11}$$

so the second index β of $\mathbf{X}^r$, the row number, is equal to the eigenvalue number i. If $D = 3$, then

$$\mathbf{X}^r = \begin{bmatrix} \hat{x}^r_{11} & \hat{x}^r_{21} & \hat{x}^r_{31} \\ \hat{x}^r_{12} & \hat{x}^r_{22} & \hat{x}^r_{32} \\ \hat{x}^r_{13} & \hat{x}^r_{23} & \hat{x}^r_{33} \end{bmatrix}. \tag{12}$$

Similarly, we define $\mathbf{X}^l$, the tensor of normalized left eigenvectors, with elements $\hat{x}^l_{i\beta}$. $\mathbf{X}^l$ is defined as

$$X^l_{\alpha\beta} = \hat{x}^l_{i\alpha}, \tag{13}$$

where

$$i = \beta. \tag{14}$$

Here also the second index β of $\mathbf{X}^l$ is equal to the eigenvalue number i. If $D = 3$, $\mathbf{X}^l$ is

$$\mathbf{X}^l = \begin{bmatrix} \hat{x}^l_{11} & \hat{x}^l_{21} & \hat{x}^l_{31} \\ \hat{x}^l_{12} & \hat{x}^l_{22} & \hat{x}^l_{32} \\ \hat{x}^l_{13} & \hat{x}^l_{23} & \hat{x}^l_{33} \end{bmatrix}. \tag{15}$$

If all the eigenvalues are all different, then, from normalization and orthogonality of the left and right eigenvectors, it follows that

$$X^r_{\gamma\alpha} X^l_{\gamma\beta} = \delta_{\alpha\beta}. \tag{16}$$

Similarly, if the eigenvalues are all different, then

$$X^r_{\gamma\alpha} X^l_{\delta\alpha} = \delta_{\gamma\delta}. \tag{17}$$

Writing Eq. (2) in terms of the normalized right eigenvectors

$$A_{\alpha\beta} \hat{x}^r_{i\beta} = \hat{x}^r_{i\alpha} \lambda_i. \tag{18}$$

It follows that

$$A_{\alpha\beta} X^r_{\beta\gamma} = X^r_{\alpha\nu} \Lambda_{\nu\gamma}, \tag{19}$$

where

$$\Lambda_{ij} = \lambda_i \delta_{ij}. \tag{20}$$

If $D = 3$, $\mathbf{\Lambda}$ is

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}. \tag{21}$$

Multiplying Eq. (19) through by $X^l_{\alpha\delta}$ gives

$$X^l_{\alpha\delta} A_{\alpha\beta} X^r_{\beta\gamma} = X^l_{\alpha\delta} X^r_{\alpha\nu} \Lambda_{\nu\gamma} \tag{22}$$

and

$$X^l_{\alpha\delta} A_{\alpha\beta} X^r_{\beta\gamma} = \Lambda_{\delta\gamma}. \tag{23}$$

This shows that $\mathbf{A}$ can be diagonalized.

Multiplying Eq. (19) through by $X^l_{\delta\gamma}$ gives

$$A_{\alpha\beta} X^r_{\beta\gamma} X^l_{\delta\gamma} = X^r_{\alpha\nu} \Lambda_{\nu\gamma} X^l_{\delta\gamma} \tag{24}$$

and

$$A_{\alpha\delta} = X^r_{\alpha\nu}\Lambda_{\nu\gamma}X^l_{\delta\gamma}. \tag{25}$$

It follows that $A_{\alpha\beta}$ may be written in terms of its eigenvalues and right and left eigenvectors as

$$A_{\alpha\beta} = \sum_i \lambda_i \hat{x}^r_{i\alpha}\hat{x}^l_{i\beta}. \tag{26}$$

The validity of Eq. (26) can be seen at once by noting that if all the eigenvalues are different, then the eigenvectors form a basis, and so any vector may be expressed in terms of these. Taking the inner product of both sides with vectors which select one element of $A_{\alpha\beta}$ demonstrates the equality element by element.

Eq. (26) is the eigenvalue decomposition of the tensor $A_{\alpha\beta}$. We note that since the eigenvectors are orthogonal, we have

$$A_{\alpha\beta}A_{\beta\alpha} = \sum_i \lambda_i^2. \tag{27}$$

We now generalize this approach to tensors of rank different from two.

III. EIGENVALUES AND EIGENTENSORS OF TENSORS OF EVEN RANK

The proposed schemes for solving the eigenvalue problem for tensors of odd and even rank differ somewhat; dealing with tensors of even rank is more straightforward. We therefore first consider tensors of even rank. Specifically, we begin by considering the standard form of the eigenvalue problem $\mathbf{Ax} = \lambda\mathbf{x}$ in the case when $\mathbf{A}$ is a tensor of rank $r = 4$. In this case, there are two ways of interpreting the eigenvalues and eigenvectors. If $\mathbf{x}$ is taken to be a vector, then each eigenvalue λ must be second rank tensor, each, in general, with D^2 elements. Alternately, if $\mathbf{x}$ is taken to be a second rank tensor, then the eigenvalues λ are simple scalars. For simplicity, conciseness and keeping a close correspondence with the usual problem for second rank tensors, we adopt the second choice.

In indicial notation, the eigenvalue problem then becomes

$$A_{\alpha\beta\gamma\delta}x^r_{\gamma\delta} = \lambda x^r_{\alpha\beta}, \tag{28}$$

where λ is a scalar eigenvalue and $x^r_{\alpha\beta}$ is the right 'eigentensor' of $A_{\alpha\beta\gamma\delta}$. This can be rearranged to read

$$(A_{\alpha\beta\gamma\delta} - \lambda\delta_{\alpha\gamma}\delta_{\beta\delta})x^r_{\gamma\delta} = 0. \tag{29}$$

This is a system of D^2 linear equations in D^2 unknowns; for a nontrivial solution to exist, the determinant must vanish.

It is useful at this point to note that the scalar (inner) product of two tensors, say $B_{\alpha\beta}$ and $C_{\gamma\delta}$, is just the scalar (inner) product of the vectors obtained by 'unfolding' the tensors. If $D = 3$, we can define, using the elements of the tensors $B_{\alpha\beta}$ and $C_{\gamma\delta}$, the vectors

$$\tilde{\mathbf{B}} = (B_{11}, B_{12}, B_{13,} B_{21}, B_{22}, B_{23}, B_{31}, B_{32}, B_{33}) \tag{30}$$

and

$$\tilde{\mathbf{C}} = (C_{11}, C_{12}, C_{13}, C_{21}, C_{22}, C_{23}, C_{31}, C_{32}, C_{33}). \tag{31}$$

Then

$$B_{\alpha\beta}C_{\alpha\beta} = \tilde{\mathbf{B}} \cdot \tilde{\mathbf{C}}. \tag{32}$$

More formally, the unfolding corresponds to replacing the pair of indices $\alpha\beta$ by a single index θ, whose value is given by

$$\theta = (\alpha - 1)D^1 + (\beta - 1)D^0 + 1, \tag{33}$$

which here can take on the values $1, 2, ... 9$. Then

$$\tilde{B}_\theta = B_{\alpha\beta} \tag{34}$$

and we can write

$$B_{\alpha\beta}C_{\alpha\beta} = \tilde{\mathbf{B}} \cdot \tilde{\mathbf{C}} = \tilde{B}_\theta \tilde{C}_\theta. \tag{35}$$

Re-labeling Eq. (33) gives

$$\phi = (\gamma - 1)D^1 + (\delta - 1)D^0 + 1, \tag{36}$$

which also takes on the values $1, 2, ... 9$. Using this notation, we can unfold the second rank tensor $x_{\alpha\beta}$ and define the vectors

$$\tilde{x}^l_\theta = x^l_{\alpha\beta} \tag{37}$$

and

$$\tilde{x}^r_\phi = x^r_{\gamma\delta}, \tag{38}$$

and unfold the fourth rank tensor $A_{\alpha\beta\gamma\delta}$ to define the second rank tensor

$$\tilde{A}_{\theta\phi} = A_{\alpha\beta\gamma\delta}. \tag{39}$$

In terms of the unfolded quantities, Eq. (28) becomes

$$\tilde{A}_{\theta\phi}\tilde{x}^r_\phi = \lambda\tilde{x}^r_\theta, \tag{40}$$

which is just the usual standard eigenvalue problem; it can be solved exactly as Eq. (2). Since Eq. (40) is the same as Eq. (28), the eigenvalues of $A_{\alpha\beta\gamma\delta}$ in Eq. (28) are the same as those of $\tilde{A}_{\theta\phi}$. The secular equation is

$$|\tilde{A}_{\theta\phi} - \lambda\delta_{\theta\phi}| = 0, \tag{41}$$

here $\tilde{A}_{\theta\phi}$ is a 9×9 matrix, therefore there are 9 eigenvalues. Once the eigenvalues are determined, the eigenvectors $\tilde{\mathbf{x}}^r_i$ corresponding to λ_i can be obtained from Eq. (40), and $\tilde{\mathbf{x}}^l_i$ can be obtained similarly. Again, left and right eigenvectors belonging to different eigenvalues are orthogonal. The eigentensors of Eq. (28) are given by

$$x^l_{i\alpha\beta} = \tilde{x}^l_{i\theta} \tag{42}$$

and

$$x^r_{i\gamma\delta} = \tilde{x}^r_{i\phi}. \tag{43}$$

The eigentensors can be normalized, as before, so that

$$\hat{x}^r_{i\gamma\delta}\hat{x}^l_{i\gamma\delta} = 1. \tag{44}$$

If all the eigenvalues are different, $\tilde{A}_{\theta\phi}$ can be written in terms of its eigentensor decomposition as

$$\tilde{A}_{\theta\phi} = \sum_i \lambda_i \frac{\tilde{x}^r_{i\theta}\tilde{x}^l_{i\phi}}{\tilde{x}^r_{i\psi}\tilde{x}^l_{i\psi}}. \tag{45}$$

Equivalently, one can write the eigenvalue expansion for $A_{\alpha\beta\gamma\delta}$,

$$A_{\alpha\beta\gamma\delta} = \sum_i \lambda_i\hat{x}^r_{i\alpha\beta}\hat{x}^l_{i\gamma\delta}, \tag{46}$$

whose validity is guaranteed by Eq. (45).

If the eigenvalues are all different, $\mathbf{A}$ can be diagonalized. It is useful to define $\mathbf{X}^r$, the fourth rank tensor of normalized right eigentensors, with elements $\hat{x}^r_{i\alpha\beta}$. $\mathbf{X}^r$ is defined as

$$X^r_{\alpha\beta\gamma\delta} = \hat{x}^r_{i\alpha\beta}, \tag{47}$$

where

$$i = (\gamma - 1)D^1 + (\delta - 1)D^0 + 1, \tag{48}$$

so the second pair of indices $\gamma\delta$ of $\mathbf{X}^r$ correspond to the eigenvalue number i. Similarly, we define $\mathbf{X}^l$, the tensor of normalized left eigentensors, with elements $\hat{x}^l_{i\delta\beta}$. $\mathbf{X}^l$ is defined as

$$X^l_{\alpha\beta\gamma\delta} = \hat{x}^l_{i\alpha\beta}, \tag{49}$$

where again

$$i = (\gamma - 1)D^1 + (\delta - 1)D^0 + 1, \tag{50}$$

so the second pair of indices $\gamma\delta$ of $\mathbf{X}^l$ correspond to the eigenvalue number i.

If the eigenvalues are all different, then, from normalization and orthogonality of the left and right eigentensors, it follows that

$$X^r_{\alpha\beta\gamma\delta} X^l_{\alpha\beta\mu\nu} = \delta_{\gamma\mu}\delta_{\delta\nu} \tag{51}$$

and

$$X^r_{\alpha\beta\gamma\delta} X^l_{\mu\nu\gamma\delta} = \delta_{\alpha\mu}\delta_{\beta\nu}. \tag{52}$$

Writing Eq. (28) in terms of the normalized right eigentensors

$$A_{\alpha\beta\sigma\tau} x^r_{i\sigma\tau} = x^r_{i\alpha\beta}\lambda_i. \tag{53}$$

It follows that

$$A_{\alpha\beta\sigma\tau} X^r_{\sigma\tau\mu\nu} = X^r_{\alpha\beta\gamma\delta}\Lambda_{\gamma\delta\mu\nu}, \tag{54}$$

where

$$\Lambda_{\gamma\delta\mu\nu} = \lambda_i \delta_{\gamma\mu}\delta_{\delta\nu} \tag{55}$$

and

$$i = (\gamma - 1)D^1 + (\delta - 1)D^0 + 1, \tag{56}$$

where $\mathbf{\Lambda}$ is the diagonal fourth rank tensor.

Multiplying Eq. (54) through by $X^l_{\alpha\beta\eta\rho}$ gives

$$X^l_{\alpha\beta\eta\rho} A_{\alpha\beta\sigma\tau} X^r_{\sigma\tau\mu\nu} = X^l_{\alpha\beta\eta\rho} X^r_{\alpha\beta\gamma\delta}\Lambda_{\gamma\delta\mu\nu} \tag{57}$$

and

$$X^l_{\alpha\beta\eta\rho} A_{\alpha\beta\sigma\tau} X^r_{\sigma\tau\mu\nu} = \Lambda_{\eta\rho\mu\nu}. \tag{58}$$

This shows that the fourth rank tensor **A** can be diagonalized.

Multiplying Eq. (54) through by $X^l_{\eta\rho\mu\nu}$ gives

$$A_{\alpha\beta\sigma\tau}X^r_{\sigma\tau\mu\nu}X^l_{\eta\rho\mu\nu} = X^r_{\alpha\beta\gamma\delta}\Lambda_{\gamma\delta\mu\nu}X^l_{\eta\rho\mu\nu}$$

and

$$A_{\alpha\beta\eta\rho} = X^r_{\alpha\beta\gamma\delta}\Lambda_{\gamma\delta\mu\nu}X^l_{\eta\rho\mu\nu}.$$

This can be written as

$$A_{\alpha\beta\eta\rho} = \sum_i \lambda_i \hat{x}^r_{i\alpha\beta}\hat{x}^l_{i\eta\rho},$$

which is the eigenvalue decomposition, as in Eq. (46).

It is straightforward to extend the above approach to tensors of arbitrary even rank. If the rank of the tensor **A** is r, then, in the eigenvalue equation $\mathbf{Ax} = \lambda\mathbf{x}$, the eigentensor **x** will have rank $r/2$. The rank $r/2$ tensor **x** can be expressed as a vector $\tilde{\mathbf{x}}$, whose index θ is given, in terms of the $r/2$ indices of **x,** by

$$\theta = (\alpha - 1)D^{r/2-1} + (\beta - 1)D^{r/2-2} + ... + (\nu - 1)D^0 + 1, \tag{59}$$

where $\alpha\beta...\nu$ are the $r/2$ indices of **x**, and D is the dimension. θ thus takes on values $1, 2, ...D^{r/2}$. The rank r tensor **A** can then be expressed as a rank 2 tensor $\tilde{\mathbf{A}}$, with the indices θ and ϕ; ϕ also takes on values $1, 2, ...D^{r/2}$. The eigenvalue equation can then be written in the usual standard form of Eq. (40). The eigenvalues of **A** are the same as those of $\tilde{\mathbf{A}}$. The secular equation is

$$|\tilde{A}_{\theta\phi} - \lambda\delta_{\theta\phi}| = 0, \tag{60}$$

here $\tilde{A}_{\theta\phi}$ is a $D^{r/2} \times D^{r/2}$ matrix, with $D^{r/2}$ eigenvalues λ_i. Once the eigenvalues are determined, the eigenvectors $\tilde{\mathbf{x}}^r_i$ corresponding to λ_i can be obtained from Eq. (40), and $\tilde{\mathbf{x}}^l_i$ can be obtained similarly. The eigentensors are given by

$$x^r_{i\alpha\beta...\nu} = \tilde{x}^r_{i\theta} \tag{61}$$

and

$$x^l_{i\alpha\beta...\nu} = \tilde{x}^l_{i\phi}. \tag{62}$$

In summary, a tensor **A** of even rank r in dimension D has $D^{r/2}$ eigenvalues, given by

$$|\tilde{A}_{\theta\phi} - \lambda\delta_{\theta\phi}| = 0, \tag{63}$$

where $\tilde{A}_{\theta\phi}$ is the unfolded representation of $\mathbf{A}$, given by

$$\tilde{A}_{\theta\phi} = A_{\alpha'\beta'...\nu'\alpha\beta...\nu}, \tag{64}$$

where

$$\theta = (\alpha' - 1)D^{r/2-1} + (\beta' - 1)D^{r/2-2} + ... + (\nu' - 1)D^0 + 1 \tag{65}$$

and

$$\phi = (\alpha - 1)D^{r/2-1} + (\beta - 1)D^{r/2-2} + ... + (\nu - 1)D^0 + 1. \tag{66}$$

The left and right eigentensors $x^l_{i\alpha'\beta'...\nu'}$ and $x^r_{i\alpha\beta...\nu}$ of $A_{\alpha'\beta'...\nu'\alpha\beta...\nu}$ can be obtained from the left and right eigenvectors $\tilde{x}^l_{i\theta}$ and $\tilde{x}^r_{i\phi}$ of $\tilde{A}_{\theta\phi}$, corresponding to the eigenvalue λ_i, from

$$x^l_{i\alpha'\beta'...\nu'} = \tilde{x}^l_{i\theta} \tag{67}$$

and

$$x^r_{i\alpha\beta...\nu} = \tilde{x}^r_{i\phi}. \tag{68}$$

If all the eigenvalues are different, then one can write explicitly

$$A_{\alpha'\beta'...\nu'\alpha\beta...\nu} = \sum_{i=1}^{D^{r/2}} \lambda_i \frac{x^r_{i\alpha'\beta'...\nu'} x^l_{i\alpha\beta...\nu}}{x^r_{i\alpha''\beta''...\nu''} x^l_{i\alpha''\beta''...\nu''}}, \tag{69}$$

which is the eigenvalue decomposition. We note here that if the rank r of $\mathbf{A}$ is of the form $r = 2^n$, where $n > 2$ is an integer, then $r/2$ is even, and the eigentensors themselves can be decomposed into their eigenvalues and eigentensors if their eigenvalues are different. This process can be continued, hierarchically, so long as all the eigenvalues are different, until $\mathbf{A}$ is written in terms of vectors of dimension D.

We note that the eigentensors are orthogonal, and therefore

$$A_{\alpha'\beta'...\nu'\alpha\beta...\nu} A_{\alpha\beta...\nu\alpha'\beta'...\nu'} = \sum_{i=1}^{D^{r/2}} \lambda_i^2. \tag{70}$$

IV. EIGENVALUES AND EIGENTENSORS OF TENSORS OF ODD RANK

We now consider the eigenvalue problem $\mathbf{Ax} = \lambda\mathbf{x}$ when $\mathbf{A}$ is a tensor of odd rank. Since both sides must have the same rank, if either inner or outer products are used, λ must have odd rank. If λ is chosen to be a vector of rank $r = 1$, then $\mathbf{x}$ must have rank $r = 2$. This leads to a system of equations which is underdetermined; there are no unique solutions for either the eigenvalues or the eigentensors. This can be seen at once if **A** is rank $r = 3$; then

$$A_{\alpha\beta\gamma}x_{\beta\gamma} = \lambda_\delta x_{\delta\alpha}, \tag{71}$$

which can be written as

$$(A_{\alpha\beta\gamma} - \lambda_\beta \delta_{\alpha\gamma})x_{\beta\gamma} = 0. \tag{72}$$

If $D = 3$, this gives 3 equations in the 9 unknowns in addition to λ_β, hence there is no solvability condition to determine λ_β. Choosing λ to have a higher rank does not resolve this problem, neither does increasing the number of dimensions D. Since there are no unique solutions for either the eigenvalues or the eigentensors if λ has odd rank, and since the requirement that λ have odd rank follows directly from **A** having odd rank and using inner or outer products, we turn to the remaining option of using the cross product. This enables the augmentation (or diminution) of the odd rank of tensor **A** via the Levi-Civita antisymmetric symbol to obtain a tensor of even rank, which has scalar eigenvalues.

We distinguish two schemes for solving the eigenvalue problem for odd rank tensors using this approach.

A. First Scheme

In the first scheme, if **A** is rank $r = 3$ and $D = 3$, the rank of **A** can be augmented by forming the rank $r = 4$ tensor

$$B_{\alpha\beta\gamma\delta} = A_{\alpha\nu\gamma}\varepsilon_{\beta\nu\delta}, \tag{73}$$

where $\varepsilon_{\beta\nu\delta}$ is the rank $r = 3$ antisymmetric symbol. If **A** is a proper tensor, then **B** is a pseudotensor. The eigenvalue problem for **B** can be solved using the method described above for tensors of even rank. In this case, there will be $D^{r/2} = 9$ eigenvalues. We note that the diagonal elements of the unfolded tensor $\tilde{\mathbf{B}}$ are zero, and $\tilde{\mathbf{B}}$ is traceless. It follows that the 9 eigenvalues are not independent, and

$$\sum_{i=1}^{9} \lambda_i = 0. \tag{74}$$

It is interesting to note that if **A** is antisymmetric in the outermost indices (that is, $A_{\alpha\nu\gamma} = -A_{\gamma\nu\alpha}$), then $\tilde{\mathbf{B}}$ is symmetric. This can be seen by considering the transpose of $\tilde{\mathbf{B}}$,

$$\tilde{B}_{\phi\theta} = B_{\gamma\delta\alpha\beta} = A^s_{\gamma\nu\alpha}\varepsilon_{\delta\nu\beta} = -A^s_{\alpha\nu\gamma}\varepsilon_{\delta\nu\beta} = A^s_{\alpha\nu\gamma}\varepsilon_{\beta\nu\delta} = B_{\alpha\beta\gamma\delta} = \tilde{B}_{\theta\phi}. \tag{75}$$

If **A** is antisymmetric in the outermost indices and real, then the eigenvalues are real and the eigenvectors are orthogonal. Conversely, if **A** is symmetric in the outermost indices, $\tilde{\mathbf{B}}$ is antisymmetric. If **A** is symmetric in the outermost indices and real, then the eigenvalues are imaginary, and the eigenvectors are orthogonal.

If the eigenvalues and eigentensors of **B** are λ and $x_{\gamma\delta}$, then

$$A_{\alpha\nu\gamma}\varepsilon_{\beta\nu\delta}x_{\gamma\delta} = \lambda x_{\alpha\beta}, \tag{76}$$

which may be understood to be of the form $\mathbf{Ax} = \lambda\mathbf{x}$, where the product of **A** and **x** consists of one inner product, and one cross product. If **A** is a proper tensor, **B** is a pseudotensor and the eigenvalues are pseudoscalars. Conversely, if **A** is a pseudotensor, **B** is a proper tensor, and the eigenvalues are proper scalars. Since if the eigenvalues are all different, **B** may be expanded in terms of its eigenvalues and eigenvectors to give

$$A_{\alpha\nu\gamma}\varepsilon_{\beta\nu\delta} = \sum_{i=1}^{9} \lambda_i \frac{x^r_{i\alpha\beta}x^l_{i\gamma\delta}}{x^l_{i\nu\mu}x^r_{i\nu\mu}}, \tag{77}$$

We note that the eigentensors $x^r_{i\alpha\beta}$ and $x^l_{j\alpha\beta}$ are orthogonal. Using the identity

$$\varepsilon_{\beta\nu\delta}\varepsilon_{\beta\eta\delta} = 2\delta_{\nu\eta}, \tag{78}$$

we obtain

$$A_{\alpha\eta\gamma} = \frac{1}{2}\sum_{i=1}^{9} \lambda_i \varepsilon_{\beta\eta\delta} \frac{x^r_{i\alpha\beta}x^l_{i\gamma\delta}}{x^l_{i\nu\mu}x^r_{i\nu\mu}}, \tag{79}$$

the eigenvalue expansion for **A**, where the products of the eigentensors of the right may be understood to consist of one outer product and one cross product.

It is interesting to note that although the eigentensors in Eq. (79) are not orthogonal, and the eigenvalues are not independent, nonetheless

$$A_{\alpha\eta\gamma}A_{\gamma\eta\alpha} = -\frac{1}{2}\sum_{i=1}^{9} \lambda_i^2. \tag{80}$$

B. Second Scheme

The second scheme consists of separating **A** into symmetric and antisymmetric parts, and solving the eigenvalue problem and carrying out the eigenvalue expansion for each. This scheme leads to fewer eigenvalues than the first. If **A** is rank $r = 3$ and $D = 3$, the rank of the symmetric part of $\mathbf{A}$ can be augmented by forming the rank $r = 4$ tensor, while the rank of the antisymmetric part can be diminished by forming a rank $r = 2$ tensor.

We separate $A_{\alpha\beta\gamma}$ into parts symmetric and antisymmetric in the outermost indices, so that

$$A_{\alpha\beta\gamma} = A^s_{\alpha\beta\gamma} + A^a_{\alpha\beta\gamma}, \tag{81}$$

where

$$A^s_{\alpha\beta\gamma} = \frac{1}{2}(A_{\alpha\beta\gamma} + A_{\gamma\beta\alpha}) \tag{82}$$

and

$$A^a_{\alpha\beta\gamma} = \frac{1}{2}(A_{\alpha\beta\gamma} - A_{\gamma\beta\alpha}). \tag{83}$$

1. The Symmetric Part

We augment the symmetric part of $\mathbf{A}$ with the Levi-Civita symbol as before, to obtain the rank 4 tensor

$$B_{\alpha\beta\gamma\delta} = A^s_{\alpha\nu\gamma}\varepsilon_{\beta\nu\delta}, \tag{84}$$

which can be unfolded into a rank 2 tensor by transforming the indices as before. That is, we define again

$$\theta = (\alpha - 1)D^1 + (\beta - 1)D^0 + 1, \tag{85}$$

where $D = 3$, and

$$\phi = (\gamma - 1)D^1 + (\delta - 1)D^0 + 1, \tag{86}$$

giving

$$\tilde{B}_{\theta\phi} = B_{\alpha\beta\gamma\delta} = A^s_{\alpha\nu\gamma}\varepsilon_{\beta\nu\delta}. \tag{87}$$

The key point here is that if $A^s_{\alpha\nu\gamma}$ is symmetric in the outermost indices, $\tilde{\mathbf{B}}$ is antisymmetric. Since $\tilde{\mathbf{B}}$ is antisymmetric, the resulting 9^{th} order secular equation is an odd function of λ, hence one eigenvalue is zero, while the rest consist of 4 pairs of positive and negative values. If $A^s_{\alpha\nu\gamma}$ is real, the eigenvalues are imaginary. We label these eigenvalues of the symmetric part of $\mathbf{A}$ as λ^s_0, λ^s_{-1}, λ^s_{+1}, λ^s_{-2}, λ^s_{+2}, ...λ^s_{+4}, where $\lambda^s_0 = 0$, and $\lambda^s_{-1} = -\lambda^s_{+1}$, etc. If all the eigenvalues are different, the eigenvalue expansion can be written as

$$A^s_{\alpha\nu\gamma}\varepsilon_{\beta\nu\delta} = \sum_{i=1}^{4} \lambda^s_{+i}\left(\frac{x^r_{+i\alpha\beta}x^l_{+i\gamma\delta}}{x^l_{+i\nu\mu}x^r_{+i\nu\mu}} - \frac{x^r_{-i\alpha\beta}x^l_{-i\gamma\delta}}{x^l_{-i\nu\mu}x^r_{-i\nu\mu}}\right), \tag{88}$$

where the eigentensors are the folded eigenvectors of $\tilde{\mathbf{B}}$, as before.

Now the eigenvalue equation for the right eigentensor $\mathbf{x}^r_{+1}$ corresponding to λ_{+1} is

$$A^s_{\alpha\nu\gamma}\varepsilon_{\beta\nu\delta}x^r_{+1\gamma\delta} = \lambda^s_{+1}x^r_{+1\alpha\beta}, \tag{89}$$

while for the left eigentensor $\mathbf{x}^l_{+1}$, it is

$$A^s_{\alpha\nu\gamma}\varepsilon_{\beta\nu\delta}x^l_{+1\alpha\beta} = \lambda^s_{+1}x^l_{+1\gamma\delta}. \tag{90}$$

Re-labeling the indices of Eq. (90) gives

$$A^s_{\gamma\nu\alpha}\varepsilon_{\delta\nu\beta}x^l_{+1\gamma\delta} = \lambda^s_{+1}x^l_{+1\alpha\beta} \tag{91}$$

and

$$A^s_{\gamma\nu\alpha}\varepsilon_{\delta\nu\beta}x^l_{+1\gamma\delta} = -A^s_{\alpha\nu\gamma}\varepsilon_{\beta\nu\delta}x^l_{+1\gamma\delta} = \lambda^s_{+1}x^l_{+1\alpha\beta}, \tag{92}$$

or

$$A^s_{\alpha\nu\gamma}\varepsilon_{\beta\nu\delta}x^l_{+1\gamma\delta} = -\lambda^s_{+1}x^l_{+1\alpha\beta} = \lambda^s_{-1}x^l_{+1\alpha\beta}, \tag{93}$$

and comparison of Eq. (89) and Eq. (93) shows that the left eigentensor associated with λ^s_{+1} equals the right eigentensor associated with λ^s_{-1}, and, in general, $\mathbf{x}^l_{+i} = \mathbf{x}^r_{-i}$. The eigenvalue expansion of Eq. (88) can therefore be written as

$$A^s_{\alpha\nu\gamma}\varepsilon_{\beta\nu\delta} = \sum_{i=1}^{4}\lambda^s_{+i}\left(\frac{x^r_{+i\alpha\beta}x^l_{+i\gamma\delta} - x^l_{+i\alpha\beta}x^r_{+i\gamma\delta}}{x^l_{+i\nu\mu}x^r_{+i\nu\mu}}\right), \tag{94}$$

where the terms in parentheses on the r.h.s. are just the antisymmetric parts of $\mathbf{x}^l\mathbf{x}^r$. The eigentensors $x^l_{+i\gamma\delta}$ and $x^r_{+j\gamma\delta}$ are orthogonal. To obtain the eigenvalue expansion of $\mathbf{A}^s$ alone, we multiply Eq. (94) by $\varepsilon_{\beta\eta\delta}$, and using the identity in Eq. (78), we get

$$A^s_{\alpha\eta\gamma} = \frac{1}{2}\sum_{i=1}^{4}\lambda^s_{+i}\varepsilon_{\beta\eta\delta}\left(\frac{x^r_{+i\alpha\beta}x^l_{+i\gamma\delta} - x^l_{+i\alpha\beta}x^r_{+i\gamma\delta}}{x^l_{+i\nu\mu}x^r_{+i\nu\mu}}\right). \tag{95}$$

Thus the eigenvalue expansion for the odd rank tensor $\mathbf{A}^s$, symmetric in the outermost indices, has $(D^{(r+1)/2} - 1)/2 = 4$ terms. We note that here also that the eigentensors are not orthogonal, however,

$$A^s_{\alpha\eta\gamma}A^s_{\gamma\eta\alpha} = -\sum_{i=1}^{4}(\lambda^s_i)^2. \tag{96}$$

2. The Antisymmetric Part

Next, we diminish the antisymmetric part of $\mathbf{A}$,

$$A^a_{\alpha\beta\gamma} = \frac{1}{2}(A_{\alpha\beta\gamma} - A_{\gamma\beta\alpha}), \tag{97}$$

with the Levi-Civita symbol to obtain the second rank tensor

$$C_{\beta\delta} = A^a_{\alpha\beta\gamma}\varepsilon_{\delta\alpha\gamma}. \tag{98}$$

The $D^{(r-1)/2} = 3$ eigenvalues and corresponding left and right eigenvectors of the second rank tensor $\mathbf{C}$ can be found in the usual way. If the eigenvalues λ^a and corresponding left and right eigenvectors $\mathbf{x}^l$ and $\mathbf{x}^r$ are known, and all the eigenvalues are different, the eigenvalue expansion is

$$A^a_{\alpha\beta\gamma}\varepsilon_{\delta\alpha\gamma} = \sum_{i=1}^{D^{(r-1)/2}} \lambda^a_i \frac{x^r_{i\beta}x^l_{i\delta}}{x^r_{i\nu}x^l_{i\nu}}, \tag{99}$$

where the eigenvectors $x^r_{i\beta}$ and $x^l_{j\beta}$ are orthogonal. To obtain the eigenvalue expansion of $\mathbf{A}^a$ alone, we multiply Eq. (99) by $\varepsilon_{\delta\eta\mu}$, and using the identity

$$\varepsilon_{\delta\eta\mu}\varepsilon_{\delta\alpha\gamma} = \delta_{\eta\alpha}\delta_{\mu\gamma} - \delta_{\eta\gamma}\delta_{\mu\alpha}, \tag{100}$$

we obtain

$$A^a_{\alpha\beta\gamma}(\delta_{\eta\alpha}\delta_{\mu\gamma} - \delta_{\eta\gamma}\delta_{\mu\alpha}) = A^a_{\eta\beta\mu} - A^a_{\mu\beta\eta} = \sum_{i=1}^{D^{(r-1)/2}} \lambda^a_i \varepsilon_{\delta\eta\mu}\frac{x^r_{i\beta}x^l_{i\delta}}{x^r_{i\nu}x^l_{i\nu}}. \tag{101}$$

Since $\mathbf{A}^a$ is antisymmetric in the outermost indices, this gives

$$A^a_{\eta\beta\mu} = \frac{1}{2}\sum_{i=1}^{D^{(r-1)/2}} \lambda^a_i \varepsilon_{\delta\eta\mu}\frac{x^r_{i\beta}x^l_{i\delta}}{x^r_{i\nu}x^l_{i\nu}}, \tag{102}$$

the eigenvalue expansion of the antisymmetric part of $\mathbf{A}$.

Thus the eigenvalue expansion for the odd rank tensor $\mathbf{A}^a$, antisymmetric in the outermost indices, has $D^{(r-1)/2} = 3$ terms. We note here also that the eigentensors are not orthogonal, however,

$$A^a_{\alpha\eta\gamma}A^a_{\gamma\eta\alpha} - 2A^a_{\alpha\alpha\gamma}A^a_{\gamma\beta\beta} = -\frac{1}{2}\sum_{i=1}^{3}(\lambda^a_i)^2. \tag{103}$$

V. SUMMARY

The eigenvalues of tensors are independent scalars which are invariant under rotations of the coordinate system, they are therefore of key importance in a variety of areas. We have presented a method for finding eigenvalues and eigenvectors and implementing eigenvalue decomposition for tensors of rank greater than two. Our method, for tensors of even rank, is a stratightforward generalization of the usual procedure for second rank tensors. Tensors of odd rank must first be converted to tensors of even rank, and, after computing the eigenvalues and eigentensors for these, converted back to odd rank again.

A tensor of even rank r in dimension D has $D^{r/2}$ eigenvalues; these can be found by unfolding the tensor to obtain a second rank tensor in $D^{r/2}$ dimensions, and solving the corresponding secular equation. The eigentensors are obtained by folding the $D^{r/2}$ corresponding eigenvectors. This method works for tensors of arbitrary (even) rank in any dimension.

For tensors of odd rank r in $3D$, there are two distinct schemes. (The eigenvalue problem of tensors of odd rank in dimensions other than 3 will be considered elsewhere.)

The first involves augmenting the tensor to rank $r+1$ by contraction with the Levi-Civita antisymmetric symbol. The eigenvalue problem for the resulting tensor can be solved using the method proposed for tensors of even rank. This scheme gives $D^{(r+1)/2}$ eigenvalues, which are not independent.

In the second scheme, a tensor of odd rank r has $(D^{(r+1)/2}-1)/2+D^{(r-1)/2}$ eigenvalues; these can be found by separating the tensor into a symmetric and antisymmetric part in the outermost indices. The symmetric part is augmented to rank $r+1$, while the antisymmetric part is diminished to rank $r-1$ by contraction with the Levi-Civita antisymmetric symbol. The eigenvalue problem for the resulting tensors can be solved using the method proposed for tensors of even rank. Since the number of eigenvalues in the first scheme are not independent and since their number is greater than that in the second, we prefer the second scheme.

The eigenvalues of proper tensors of odd rank are pseudoscalars, and eigenvalues of pseudotensors of odd rank are proper scalars.

Tensors of odd rank which are symmetric in the outermost indices and which are real, such as the order parameter tensor of tetrahedral particles, have eigenvalues which are imaginary.

Our proposed scheme enables the determination of the eigenvalues and implementing eigenvalue decompositions of order parameter tensors of arbitrary rank, and may be useful in developing statistical theories of condensed matter systems consisting of particles with high symmetry.

APPENDIX A: EXAMPLE OF THIRD RANK TENSOR DECOMPOSITION:

Here we demonstrate decomposition of a third rank tensor with randomly chosen elements according to scheme 2.

We consider the third rank tensor $\mathbf{A}$ generated using random integers.

$$A_{1\alpha\beta} = \begin{bmatrix} -8 & 8 & -5 \\ 8 & -9 & -6 \\ 10 & -8 & 9 \end{bmatrix}, \; A_{2\alpha\beta} = \begin{bmatrix} -5 & 12 & -9 \\ -12 & 4 & -11 \\ -8 & -10 & 9 \end{bmatrix}, \; A_{3\alpha\beta} = \begin{bmatrix} 5 & -12 & -3 \\ 8 & 7 & -6 \\ -9 & -10 & 11 \end{bmatrix}. \tag{A1}$$

$$\mathbf{A}^s = \left[\begin{array}{ccc|ccc|ccc} -8 & 1.5 & 0 & 1.5 & 12 & -7.5 & 0 & -7.5 & 0 \\ 8 & -10.5 & 1 & -10.5 & 4 & -2 & 1 & -2 & -6 \\ 10 & -8 & 0 & -8 & -10 & -0.5 & 0 & -0.5 & 11 \end{array}\right]. \tag{A2}$$

$$\lambda_1^s = 24.5766i, \tag{A3}$$

$$\left[\begin{array}{ccc|ccc|ccc} 0.1297i & 0.0304i & -.0510i & 0.0304i & -.3578i & 0.1522i & -.0510i & 0.1522i & -.0597i \\ -.2862i & 0.3080i & -.0767i & 0.3080i & -.2997i & 0.0668i & -.0767i & 0.0668i & 0.0039i \\ -.2493i & 0.1637i & -.0106i & 0.1637i & 0.1049i & 0.0110i & -.0106i & 0.0110i & -.0437i \end{array}\right], \tag{A4}$$

$$\lambda_2^s = 18.5853i, \tag{A5}$$

$$\left[\begin{array}{ccc|ccc|ccc} 0.2264i & -.1282i & 0.0686i & -.1282i & -.1813i & 0.1426i & 0.0686i & 0.1426i & -.0370i \\ -.0412i & 0.1591i & 0.0323i & 0.1591i & 0.1607i & -.0555i & 0.0323i & -.0555i & 0.0736i \\ -.1882i & 0.1704i & -.1139i & 0.1704i & 0.3548i & 0.0463i & -.1139i & 0.0463i & -.0627i \end{array}\right], \tag{A6}$$

$$\lambda_3^s = 12.4244i, \tag{A7}$$

$$\left[\begin{array}{ccc|ccc|ccc} 0.0220i & -.0026i & -.0026i & -.0026i & -.0215i & 0.0347i & -.0026i & 0.0347i & 0.2106i \\ -.0139i & -.0341i & -.0193i & -.0341i & -.0304i & 0.0605i & -.0193i & 0.0605i & 0.2894i \\ -.0678i & 0.0445i & 0.1547i & 0.0445i & 0.0564i & -.0937i & 0.1547i & -.0937i & -.7334i \end{array}\right], \tag{A8}$$

$$\lambda_4^s = 4.1483i, \tag{A9}$$

$$\left[\begin{array}{ccc|ccc|ccc} 0.0798i & 0.0405i & 0.0026i & 0.0405i & 0.1035i & 0.1633i & 0.0026i & 0.1633i & -.1114i \\ -.0063i & 0.0959i & 0.1263i & 0.0959i & 0.1825i & 0.1537i & 0.1263i & 0.1537i & 0.2268i \\ 0.1124i & 0.0619i & 0.1098i & 0.0619i & 0.0305i & 0.1284i & 0.1098i & 0.1284i & 0.0848i \end{array}\right], \tag{A10}$$

$$\mathbf{A}^a = \left[\begin{array}{ccc|ccc|ccc} 0. & 6.5 & -5. & -6.5 & 0. & 4.5 & 5. & -4.5 & 0. \\ 0. & 1.5 & -7. & -1.5 & 0. & -9. & 7. & 9. & 0. \\ 0. & 0. & 9. & 0. & 0. & 9.5 & -9. & -9.5 & 0. \end{array}\right], \tag{A11}$$

$$\lambda_1^a = 21.6430, \tag{A12}$$

$$\left[\begin{array}{ccc|ccc|ccc} 0. & 0.1237 & 0.0168 & -0.1237 & 0. & 0.2099 & -0.0168 & -0.2099 & 0. \\ 0. & -0.1876 & -0.0255 & 0.1876 & 0. & -0.3181 & 0.0255 & 0.3181 & 0. \\ 0. & 0.2646 & 0.0359 & -0.2646 & 0. & 0.4489 & -0.0359 & -0.4489 & 0. \end{array}\right], \tag{A13}$$

$$\lambda_2^a = 0.6785 + 9.1197i, \tag{A14}$$

$$\left[\begin{array}{ccc|ccc|ccc} 0. & -.0619 - .2141i & -.0084 + 0.2934i & 0.0619 + 0.2141i & 0. & 0.1451 + 0.0131i & 0.0084 - .2934i & -.1451 - .0131i & 0. \\ 0. & 0.0938 - .2978i & -.2373 + 0.3359i & -.0938 + 0.2978i & 0. & 0.1591 + 0.1278i & 0.2373 - .3359i & -.1591 - .1278i & 0. \\ 0. & 0.1177 + 0.3228i & -.0180 - .4521i & -.1177 - .3228i & 0. & -.2244 - .0049i & 0.0180 + 0.4521i & 0.2244 + 0.0049i & 0. \end{array}\right], \tag{A15}$$

$$\lambda_3^a = 0.6785 - 9.1197i, \tag{A16}$$

$$\left[\begin{array}{ccc|ccc|ccc} 0. & -.0619 + 0.2141i & -.0084 - .2934i & 0.0619 - .2141i & 0. & 0.1451 - .0131i & 0.0084 + 0.2934i & -.1451 + 0.0131i & 0. \\ 0. & 0.0938 + 0.2978i & -.2373 - .3359i & -.0938 - .2978i & 0. & 0.1591 - .1278i & 0.2373 + 0.3359i & -.1591 + 0.1278i & 0. \\ 0. & 0.1177 - .3228i & -.0180 + 0.4521i & -.1177 + 0.3228i & 0. & -.2244 + 0.0049i & 0.0180 - .4521i & 0.2244 - .0049i & 0. \end{array}\right]. \tag{A17}$$

APPENDIX B: EXAMPLE OF FIRST RANK TENSOR DECOMPOSITION

The procedure proposed above for the decomposition of a third rank tensor must also work for odd ranks in general, and for a first rank tensor - a vector - in particular.

We therefore consider the problem of decomposing a first rank tensor

$$\mathbf{A} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}. \tag{B1}$$

The eigenvalue problem becomes

$$A_\alpha \varepsilon_{\alpha\beta\gamma} x_\gamma = \lambda x_\beta, \tag{B2}$$

which can be written as

$$(A_\alpha \varepsilon_{\alpha\beta\gamma} - \lambda\delta_{\beta\gamma}) x_\gamma = 0, \tag{B3}$$

and we require that

$$|A_\alpha \varepsilon_{\alpha\beta\gamma} - \lambda\delta_{\beta\gamma}| = 0, \tag{B4}$$

and the secular equation becomes

$$\lambda^3 + \lambda(a^2 + b^2 + c^2) = 0. \tag{B5}$$

The eigenvalues are therefore

$$\lambda = 0 \tag{B6}$$

and

$$\lambda = \pm i\sqrt{a^2 + b^2 + c^2} = \pm iA. \tag{B7}$$

We next determine the eigenvectors associated with these. We must have

$$\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}, \tag{B8}$$

where we have arbitrarily set $z = 1$. Since the vector is on the right of the matrix, we call these the right eigenvectors. This gives

$$x = \frac{ac - \lambda b}{c^2 + \lambda^2} \tag{B9}$$

and

$$y = \frac{bc + \lambda a}{c^2 + \lambda^2}. \tag{B10}$$

The right eigenvector associated with $\lambda = 0$ is

$$\mathbf{x}_0^r = \begin{bmatrix} a/c \\ b/c \\ 1 \end{bmatrix}, \tag{B11}$$

that is, the vector $\mathbf{A}$ itself.

The right eigenvector associated with $\lambda_+ = +i\sqrt{a^2 + b^2 + c^2}$ can be written as

$$\mathbf{x}_+^r = \begin{bmatrix} ac - ib\sqrt{a^2 + b^2 + c^2} \\ bc + ai\sqrt{a^2 + b^2 + c^2} \\ -(a^2 + b^2) \end{bmatrix}, \tag{B12}$$

and associated with $\lambda_- = -i\sqrt{a^2 + b^2 + c^2}$, it is

$$\mathbf{x}_-^r = \begin{bmatrix} ac + ib\sqrt{a^2 + b^2 + c^2} \\ bc - ai\sqrt{a^2 + b^2 + c^2} \\ -(a^2 + b^2) \end{bmatrix}. \tag{B13}$$

It is interesting to note that these vectors are complex conjugates, and that they are self-orthogonal; that is,

$$\mathbf{x}_+^r \cdot \mathbf{x}_+^r = \mathbf{x}_-^r \cdot \mathbf{x}_-^r = 0. \tag{B14}$$

but they are not orthogonal with each other,

$$\mathbf{x}_+^r \cdot \mathbf{x}_-^r = 2(a^2 + b^2)(a^2 + b^2 + c^2). \tag{B15}$$

We note that the equation $\mathbf{A} \times \mathbf{B} = \lambda\mathbf{B}$ has a non-trivial solution for $\mathbf{B}$ if $\mathbf{B} \cdot \mathbf{B} = 0$.

It is also interesting to note that if we write

$$a = r \sin\theta \cos\phi, \tag{B16}$$

$$b = r \sin\theta \sin\phi, \tag{B17}$$

and

$$c = r \cos\theta, \tag{B18}$$

then

$$\mathbf{x}^r_+ = \begin{bmatrix} ac - ib\sqrt{a^2+b^2+c^2} \\ bc + ai\sqrt{a^2+b^2+c^2} \\ -(a^2+b^2) \end{bmatrix} = r^2 \begin{bmatrix} \sin\theta\cos\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\theta\sin\phi + i\sin\theta\cos\phi \\ -\sin^2\theta \end{bmatrix}. \tag{B19}$$

This means that each term is divisible by

$$r\sin\theta = \sqrt{a^2+b^2}. \tag{B20}$$

We can also look at the left eigenvectors; this are solutions of

$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} = \lambda \begin{bmatrix} x & y & 1 \end{bmatrix}. \tag{B21}$$

These are

$$x = \frac{ac + \lambda b}{c^2 + \lambda^2} \tag{B22}$$

and

$$y = \frac{bc - \lambda a}{c^2 + \lambda^2}. \tag{B23}$$

and so the left eigenvector associated with $\lambda_+ = +i\sqrt{a^2+b^2+c^2}$ is

$$\mathbf{x}^l_+ = \begin{bmatrix} ac + ib\sqrt{a^2+b^2+c^2} \\ bc - ai\sqrt{a^2+b^2+c^2} \\ -(a^2+b^2) \end{bmatrix}, \tag{B24}$$

and associated with $\lambda_- = -i\sqrt{a^2+b^2+c^2}$, it is

$$\mathbf{x}^l_- = \begin{bmatrix} ac - ib\sqrt{a^2+b^2+c^2} \\ bc + ai\sqrt{a^2+b^2+c^2} \\ -(a^2+b^2) \end{bmatrix}. \tag{B25}$$

Again, these are self-orthogonal, and their inner product is

$$\mathbf{x}^l_+ \cdot \mathbf{x}^l_- = 2(a^2+b^2)(a^2+b^2+c^2). \tag{B26}$$

So we have

$$\mathbf{x}^r_+ = \mathbf{x}^l_- \tag{B27}$$

and

$$\mathbf{x}_-^r = \mathbf{x}_+^l. \tag{B28}$$

It is worth noting that $\lambda_+ = -\lambda_-$, but $\lambda_+ \neq \lambda_-^*$, since a, b, c may be complex.

We also note that

$$\mathbf{x}_+^l \times \mathbf{x}_+^r = \begin{bmatrix} (+i2a\sqrt{a^2+b^2+c^2})(a^2+b^2) \\ (+i2b\sqrt{a^2+b^2+c^2})(a^2+b^2) \\ (+i2c\sqrt{a^2+b^2+c^2})(a^2+b^2) \end{bmatrix}. \tag{B29}$$

If we normalize the eigenvectors so that

$$\mathbf{x}_+^l \cdot \mathbf{x}_+^r = 1, \tag{B30}$$

then

$$\mathbf{A} = -\lambda_+(\mathbf{x}_+^l \times \mathbf{x}_+^r). \tag{B31}$$

as expected. We then have the standard form

$$\mathbf{A} = \frac{1}{2}\{\lambda_+(\mathbf{x}_+^l \times \mathbf{x}_+^r) + \lambda_-(\mathbf{x}_-^l \times \mathbf{x}_-^r) + \lambda_0(\mathbf{x}_0^l \times \mathbf{x}_0^r)\} = \frac{1}{2}(A\hat{\mathbf{A}} + A\hat{\mathbf{A}}). \tag{B32}$$

and our scheme works even for a rank 1 tensor.

Acknowledgement *We are grateful to Kaushik Bhattacharya, Sharon Glotzer, Rolfe Petschek, Michael Shelley and Bahman Taheri for useful discussions. We acknowledge support from the NSF under SGER 0440299 for portions of this work. (Random integers for Appendix 1. were generated by Nicole P-M, throwing dice for the number, and flipping a coin for the sign on 12/27/05).*

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