University of Strathclyde

Periodic Disturbances in Cylindrically layered Smectic A

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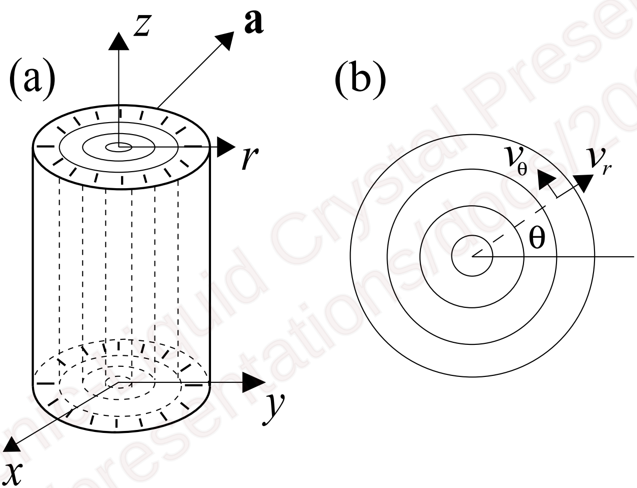
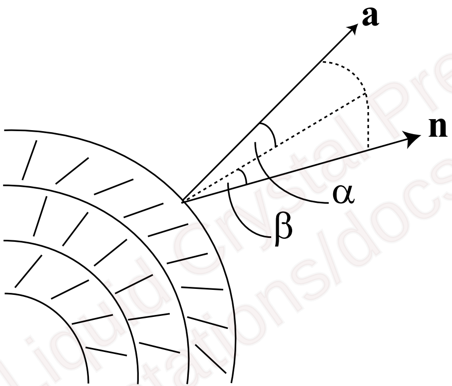
Ferroelectric Phenomena in Liquid Crystals
Kent State University
June 2007

- Dynamics of a cylindrically layered smectic A liquid crystal are studied.
- Aim to see how the sample behaves under the influence of a sinusoidal perturbation.
- The possible decoupling of the unit vectors **a** and **n** shall be considered.
- Initial investigation made by Kléman and Parodi [1]


(a)
z
a
r
y
x
(b)
v_θ
v_r
θ



- The layers are characterised by the ‘smectic layer function’ $\Phi(r, \theta, z, t)$, the level sets of $\Phi = 0$ correspond to the layer structure.

- The dynamic equations of Stewart [2] shall be used:

$$a_i = \frac{\Phi_{,i}}{|\nabla \Phi|}, \tag{1}$$

$$a_i a_i = 1, \quad n_i n_i = 1, \tag{2}$$

$$v_{i,i} = 0, \tag{3}$$

$$\rho \dot{v}_i = -\tilde{p}_{,i} + \tilde{g}_j n_{j,i} + |\nabla \Phi| a_i J_{j,j} + \tilde{t}_{ij,j}, \tag{4}$$

$$\dot{\Phi} = -\lambda_p J_{i,i}, \tag{5}$$

$$\left(\frac{\partial w_A}{\partial n_{i,j}} \right)_{,j} - \frac{\partial w_A}{\partial n_i} + \tilde{g}_i = \lambda n_i. \tag{6}$$

- We assume the smectic layer function to be of the form

$$\Phi(r,z,t) = r + u(r,z,t).$$

- Motivated by physics, the energy density function shall take the form

$$\begin{aligned} w_A &= \tfrac{1}{2}K_1^n\left(\nabla\cdot\mathbf{n}\right)^2 + \tfrac{1}{2}K_1^a\left(\nabla\cdot\mathbf{a}\right)^2 \\ &\quad + \tfrac{1}{2}B_0\left(|\nabla\Phi| + \mathbf{n}\cdot\mathbf{a} - 2\right)^2 + \tfrac{1}{2}B_1\left(1 - (\mathbf{n}\cdot\mathbf{a})^2\right). \end{aligned}$$

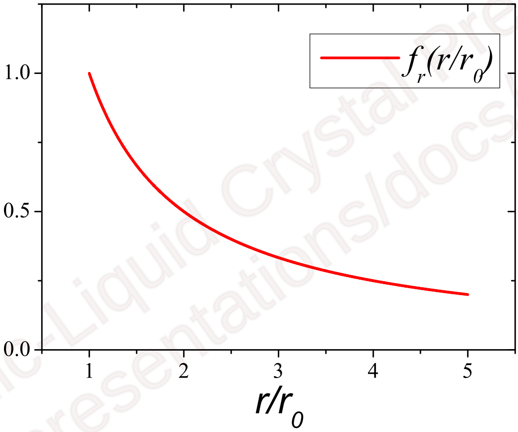
- We reduce the viscous stress tensor to

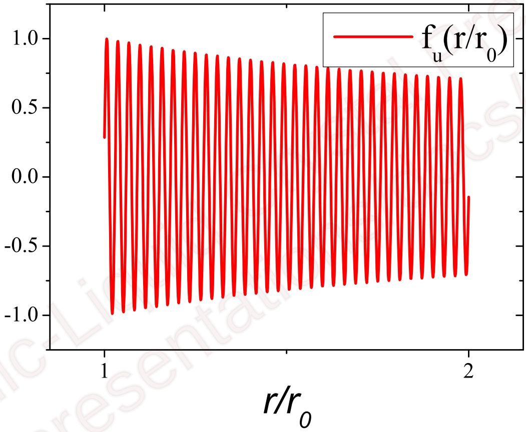
$$\tilde{t}_{ij} = \tfrac{1}{2}\mu_0\left(v_{i,j} + v_{j,i}\right).$$

- We consider first the case $\mathbf{a} \equiv \mathbf{n}$.
- Motivated by Vanaparthy *et al.* [3] and Payr *et al* [4] we assume perturbations of the form

$$\begin{pmatrix} v_r \\ v_z \\ p \\ u \end{pmatrix} (r, z, t) = \begin{pmatrix} h_r R f_r(r) \\ h_z Z f_z(r) \\ P f_p(r) \\ U f_u(r) \end{pmatrix} e^{-\omega t + iqz}.$$

- Primarily deal only with functions that depend on r and t.
- Dynamic equations result in three differential equations with three unknown quantities.
- $f_r(r) = r_0/r$
- $f_u(r) = f_1(J_i(r), Y_i(r), H_i(r))$
- $f_p(r) = f_2(\ln(r), J_i(r), Y_i(r), H_i(r))$
- However, no information on the form of parameter ω is found.


$f_r(r/r_0)$
1.0
0.5
0.0
1
2
3
4
5
r/r_0



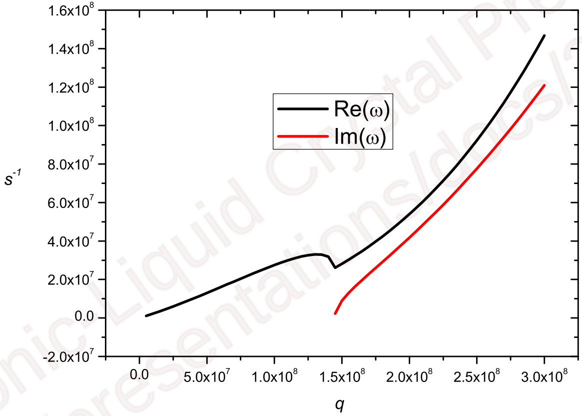
1.0
0.5
0.0
-0.5
-1.0
1
2
r/r0
fu(r/r0)

- Radial and Zenithal flow considered.
- Dynamic equations result in four differential equations with four unknown quantities.
- Equations can be manipulated into one fifth order O.D.E. in layer displacement function f_u.
- Series solution is found of the form

$$f_u(r) = \sum_{i=0}^{n} \left(\mathcal{A}_i r^{-\phi} + \mathcal{B}_i r^{\phi} + \mathcal{C}_i \ln r + \mathcal{D}_i \right) r^i,$$

where $\phi = q\sqrt{-K_1^a/B_0}$ and $\mathcal{A}_i$, $\mathcal{B}_i$, $\mathcal{C}_i$ and $\mathcal{D}_i$ are lengthy expressions involving physical parameters.

- Back-substitution provides solutions for f_r, f_z and f_p.
- Final dynamic equation provides us with a defining equation for ω in terms of q.
- Real and positive solutions for ω are found with bounds for wave number q.


1.6x10^8
1.4x10^8
1.2x10^8
1.0x10^8
8.0x10^7
6.0x10^7
4.0x10^7
2.0x10^7
0.0
-2.0x10^7
s^-1
Re(ω)
Im(ω)
0.0
5.0x10^7
1.0x10^8
1.5x10^8
2.0x10^8
2.5x10^8
3.0x10^8
q

- We now allow the layer normal and director to decouple.
- Assume perturbations of the form

$$\begin{pmatrix} v_r \\ v_z \\ p \\ u \\ \alpha \\ \beta \end{pmatrix}(r,z,t) = \begin{pmatrix} h_r R f_r(r) \\ h_z Z f_z(r) \\ P f_p(r) \\ U f_u(r) \\ \bar{\alpha} f_\alpha(r) \\ \bar{\beta} f_\beta(r) \end{pmatrix} e^{-\omega t + iqz}.$$

- To first order in dynamic equations, we are forced into constraint $f_\alpha \equiv 0$.

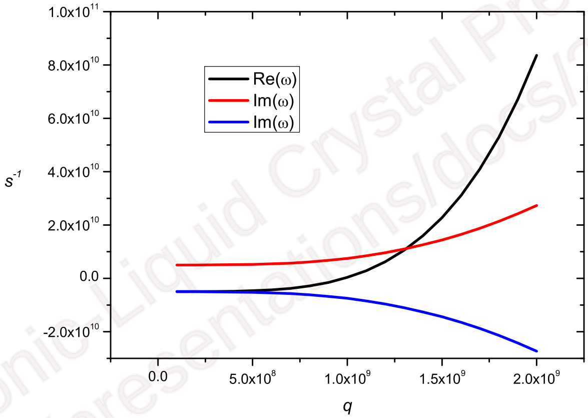
- Primarily deal only with functions that depend on r and t.
- Dynamic equations result in four differential equations with four unknown quantities.
- $f_\beta \equiv 0$
- Solutions exactly match for the case $\mathbf{n} \equiv \mathbf{a}$.

- Radial and Zenithal flow considered.
- Dynamic equations result in five differential equations with five unknown quantities.
- Equations can be manipulated into one sixth order O.D.E. in angle function f_β.
- Series solution is found of the form

$$f_\beta(r) = \sum_{i=0}^{n} \left(\mathcal{A}_i r^{-\phi} + \mathcal{B}_i r^{\phi} + \mathcal{C}_i \ln r + \mathcal{D}_i \right) r^i,$$

where $\phi = \sqrt{-K_1^a / B_0}$ and $\mathcal{A}_i$, $\mathcal{B}_i$, $\mathcal{C}_i$ and $\mathcal{D}_i$ are lengthy expressions involving physical parameters.

- Back-substitution provides solutions for f_u, f_r, f_z and f_p.
- Final dynamic equation provides us with a defining equation for ω in terms of q.
- Complex solutions exist for ω, for all values of wavenumber q.


1.0x10^11
8.0x10^10
6.0x10^10
4.0x10^10
2.0x10^10
0.0
-2.0x10^10
s^-1
Re(ω)
Im(ω)
Im(ω)
0.0
5.0x10^8
1.0x10^9
1.5x10^9
2.0x10^9
q

- The dynamic equations of a cylindrically layered SmA liquid crystal sample under the influence of a sinusoidal perturbation are considered.
- General ansatze are introduced for the velocity, pressure, layer displacement and size of the angles between the layer normal and director.
- Stability can be shown in the radial velocity only case.
- Lower bound for wave number found when zenithal velocity is considered.
- Work could be extended with an expansion of the viscous stress tensor used and inclusion of the azimuthal velocity v_θ.

- Analysis of SmA under Couette and Poiseuille flow with decoupled layer normal and director.

- Non-linear analysis of smectic layer function Φ, in planar and cylindrical geometries.

Thanks to

- Liquid Crystal Institute and Department of Mathematics, Kent State University
- The British Liquid Crystal Society
- Department of Mathematics, University of Strathclyde

Thank you for your attention.

[1] M. Kléman and O. Parodi, *J. Phys. (Paris)* **35**, 671 (1975).

[2] I. W. Stewart, *Continuum Mech. Thermodyn.*, **18**, 343 (2007).

[3] S. H. Vanaparthy, E. Meiburg and D. Wilhelm, *J. Fluid Mech.* **497**, 99 (2003).

[4] M. Payr, S. H. Vanaparthy and E. Meiburg, *J. Fluid Mech.* **525**, 333 (2005).