Ward Identities and Semisoft Elasticity

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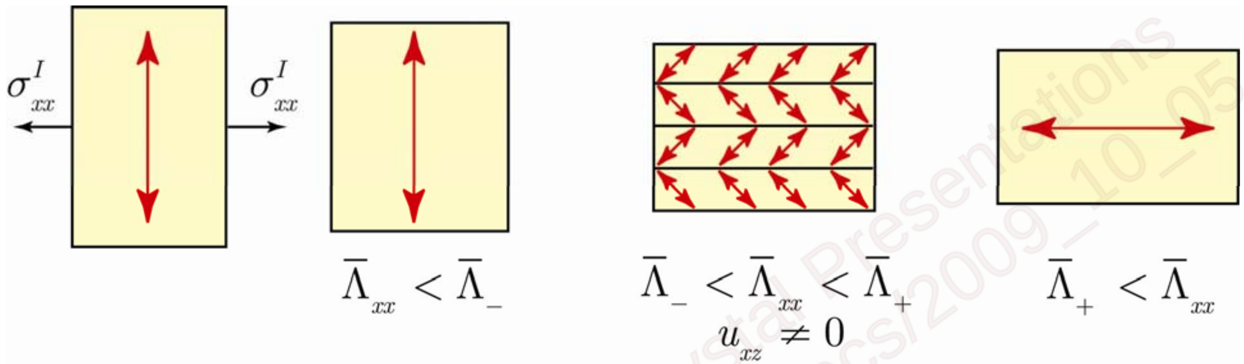
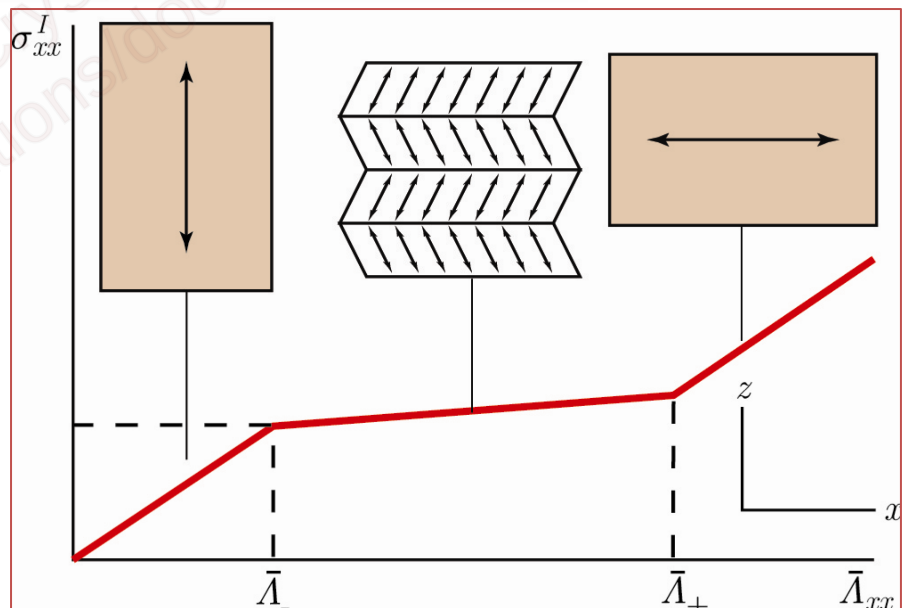
Semisoft Elastic Response



Λ_- and Λ_+ signal phase transitions at which u_{xz}, respectively, becomes nonzero and vanishes



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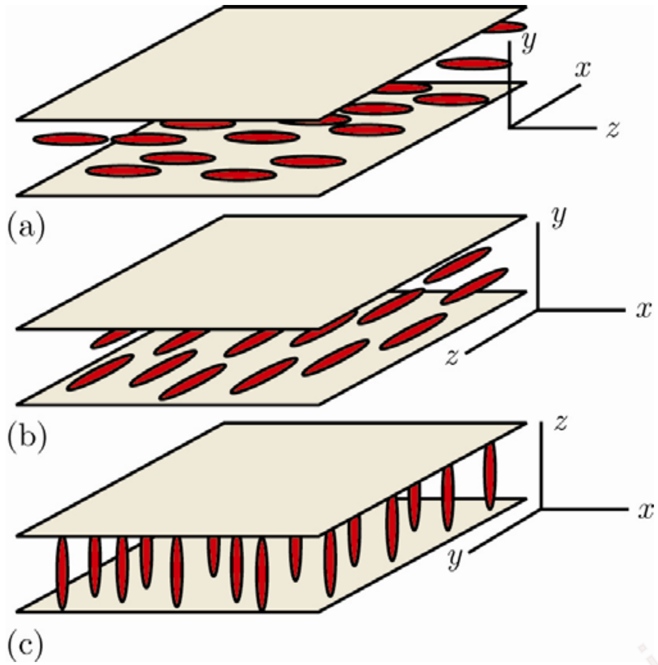
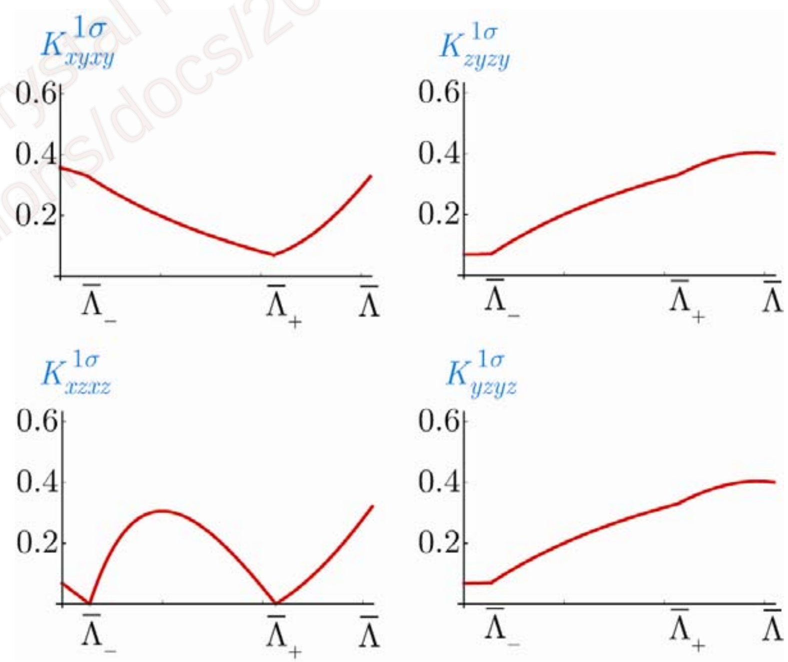
Differential Elastic Response



Displacement of the top plate relative to the bottom one produces xy and zy stress in (a) and (b) and xz and yz stresses in (c).

Forms of elastic tensors independent of detailed model: follow from Ward identities.



$$\delta\Lambda'_{ij} = K^{\sigma}_{ijkl}\delta\sigma^{I}_{kl}$$

$$(c)\ \delta\Lambda'_{zx} = \delta\Lambda'_{zy} = \delta\Lambda'_{xy} = \delta\Lambda'_{yx} = 0$$

$$(a)\ \delta\Lambda'_{yx} = \delta\Lambda'_{yz} = \delta\Lambda'_{xz} = \delta\Lambda'_{zx} = 0$$

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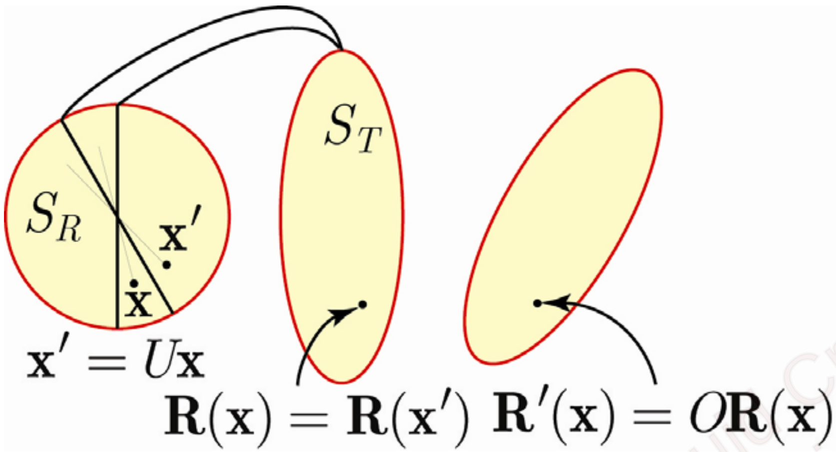
Deformations and Strains



$$\mathbf{R}(\mathbf{x}) = \mathbf{x} + \mathbf{u}(\mathbf{x})$$

$$\Lambda_{i\alpha} = \frac{\partial R_i}{\partial x_\alpha} = \delta_{i\alpha} + \eta_{i\alpha}$$

$$\eta_{i\alpha} = \partial_\alpha u_i$$

$$dR^2 - dx^2 = 2u_{\alpha\beta}dx_\alpha dx_\beta$$

$$\underline{u} = \tfrac{1}{2}(\underline{\Lambda}^T \underline{\Lambda} - \underline{\delta}) \approx \tfrac{1}{2}(\underline{\eta} + \underline{\eta}^T)$$

$$u_{\alpha\beta} = \tfrac{1}{2}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha + \partial_\alpha u_k \partial_\beta u_k\right)$$

$u_{\alpha\beta}$ is invariant under rotations in target space – used to describe systems without aligning stresses

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Elastic Energy

Decompose free energy into an isotropic part invariant under rotations in S_R (i.e. of $\mathbf{x}$) and an anisotropic part (Assume Maier Saupe order parameter Q_{ij} has been integrated out).

$$f(\underline{u}) = f_{\mathrm{iso}}(\underline{u}) + f_{\mathrm{ani}}(\underline{u})$$

$$f_{\mathrm{ani}}(\underline{u}) = -hu_{zz}$$

Add "internal" or second PK stress: still invariant w.r.t. to rotations in S_T

$$g(\underline{u}) = f(\underline{u}) - \sigma_{ij} u_{ij}$$

New equilibrium deformation and strain

$$u^0_{ij} = \frac{1}{2}\left(\Lambda^0_{ki}\Lambda^0_{kj} - \delta_{ij}\right)$$

$$\Lambda^0_{ij} = \begin{pmatrix} \Lambda^0_{xx} & 0 & \Lambda^0_{xz} \\ 0 & \Lambda^0_{yy} & 0 \\ 0 & 0 & \Lambda^0_{zz} \end{pmatrix}$$

Elastic Tensor

Energy of "harmonic" deviations from equilibrium

$$\delta g = \frac{1}{2} C_{ijkl} \delta u_{ij} \delta u_{kl}$$

New reference position

$$x'_i = \Lambda^0_{ij} x_j ; \quad \Lambda_{ik} = \Lambda'_{ik} \Lambda^0_{kj}$$

Strain w.r.t. to new reference pos.

$$\delta \underline{u} = \underline{\Lambda}^{0T} \cdot \underline{u}' \cdot \underline{\Lambda}$$

$$\delta g' = \frac{1}{2} C'_{ijkl} \delta u'_{ij} \delta u'_{kl}$$

Elastic tensor w.r.t to new reference space – symmetric under interchange of i and j and k and l

$$C'_{ijkl} = \frac{1}{\det \underline{\Lambda}^0} \Lambda^0_{ip} \Lambda^0_{kr} C_{pqrs} \Lambda^{0T}_{qj} \Lambda^{0T}_{sl}$$

Elastic Tensor for Deformations

External 1st PK stress induces deformation: Defines a direction in S_T: σ^I_{xx} is the force along x in the target space per unit area of the reference space

$$g^I = f(\underline{u}) - \sigma^I_{ij}\Lambda_{ij}$$

First PK stress tensor (in equilibrium)

$$\sigma^I_{ij} = \frac{\partial f}{\partial \Lambda_{ij}} = \Lambda_{ik}\frac{\partial f}{\partial u_{kj}}$$

$$K_{ijkl} = \left.\frac{\partial^2 f}{\partial \Lambda_{ij}\partial \Lambda_{kl}}\right|_{\underline{\Lambda}=\underline{\Lambda}^0} = \delta_{ik}\sigma_{jl} + \Lambda^0_{ip}\Lambda^0_{kr}C_{pjrl}$$

$$\sigma_{ij} = (\Lambda^0)^{-1}_{ik}\sigma^I_{kj}$$

Deformation Tensor Relative to New State

$$\Lambda_{ik} = \Lambda'_{ik} \Lambda^0_{kj}$$

$$\delta g'^I = \frac{1}{2} K'_{ijkl} \delta\Lambda'_{ij} \delta\Lambda'_{kl}; \quad K'_{ijkl} = \delta_{ik} \sigma'_{jl} + C'_{ijkl}$$

$$\sigma'_{jl} = \frac{1}{\det \underline{\Lambda}^0} \Lambda^0_{jp} \sigma_{qs} \Lambda^{0T}_{sl}$$

Cauchy Stress Tensor

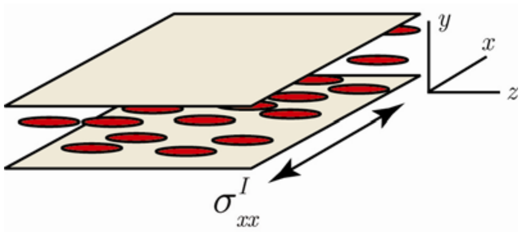
K'_{ijkl} has a symmetric part C'_{ijkl} and an anti-symmetric part, which defines preferred direction in S_T

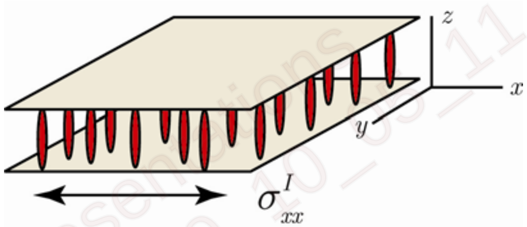
Small Deformations:

$$\delta g'^I = \frac{1}{2}[\sigma'_{xx}(\delta\theta_y - \varepsilon_{xz})^2 + 4C'_{xzxz}\varepsilon^2_{xz}]$$

$$\delta\theta_y = \frac{1}{2}(\Lambda'_{xz} - \Lambda'_{zx}); \quad \varepsilon_{xz} = \frac{1}{2}(\Lambda'_{xz} + \Lambda'_{zx})$$

Elastic Moduli at "Constant Stress"





$$\delta\Lambda'_{yx} = \delta\Lambda'_{yz} = \delta\Lambda'_{xz} = \delta\Lambda'_{zx} = 0$$
$$\delta\sigma_{xy} \text{ or } \delta\sigma_{zy} \neq 0; \text{ and}$$
$$\delta\sigma_{xx} = \delta\sigma_{yy} = \delta\sigma_{zz} = 0$$

$$\delta\Lambda'_{zx} = \delta\Lambda'_{zy} = \delta\Lambda'_{xy} = \delta\Lambda'_{yx} = 0$$
$$\delta\sigma_{xz} \text{ or } \delta\sigma_{yz} \neq 0; \text{ and}$$
$$\delta\sigma_{xx} = \delta\sigma_{yy} = \delta\sigma_{zz} = 0$$

$$xx = 1;\ yy = 2;\ zz = 3;$$
$$xz = 4;\ zy = 5;\ xy = 6.$$

$$K'_{ab}\delta\Lambda'_b = \delta\sigma'^I_a$$
$$\delta\Lambda'_a = S'_{ab}\delta\sigma'^I_b; \qquad \underline{S}' = \underline{K}'^{-1}$$

$$K^\sigma_{aa} = \frac{\partial\sigma'^I_a}{\partial\Lambda'_b} = \frac{1}{S'_{aa}}$$

Ward Identities for C'_{ijkl}

$$f(\underline{u}) = f_{\text{iso}}(\underline{u}) - -hu_{zz}$$

$f_{\text{iso}}(\underline{u})$ is invariant under $\underline{u} \rightarrow \underline{\underline{U}}\,\underline{u}\,\underline{U}^{-1}$

$$U^a_{ij} = \delta_{ij} - \theta_a \varepsilon_{ajk}$$

$$u_{ij} \rightarrow u_{ij} + \theta_a(\varepsilon_{aki}u_{kj} + \varepsilon_{akj}u_{ik})$$

$$\frac{df}{d\theta_a} = \frac{\partial f}{\partial u_{ij}}\frac{du_{ij}}{d\theta_a} = \frac{df_{\text{ani}}}{d\theta_a} = -h_{ij}\frac{du_{ij}}{d\theta_a}$$

$$(\varepsilon_{ari}u_{rj} + \varepsilon_{arj}u_{ir})(\sigma_{ij} + h_{ij}) = 0$$

These conditions must hold:

$$\begin{aligned} 2u_{yz}h &= 0, \quad a = x; \\ 2u_{xz}(h - \sigma_{xx}) &= 0, \quad a = y; \\ 2u_{xy}\sigma_{xx} &= 0, \quad a = z. \end{aligned}$$

Hence

$$u_{xz} = 0 \text{ or } h = \sigma_{xx}$$

Ward Identities for C_{ijkl}

$$\frac{\partial}{\partial u_{kl}}(\varepsilon_{ari} u_{rj} + \varepsilon_{arj} u_{ir})(\sigma_{ij} + h_{ij}) = 0$$

$$2\varepsilon_{ari} u_{rj} C_{ijkl} + \varepsilon_{aki}(\sigma_{il} + h_{il}) + \varepsilon_{ali}(\sigma_{ik} + h_{ik}) = 0$$

$$u_{xz} = 0$$

$$C_{xyxy} = \frac{1}{2}\frac{\sigma_{xx}}{u_{xx} - u_{yy}};$$

$$C_{yzyz} = \frac{1}{2}\frac{h}{u_{zz} - u_{yy}};$$

$$C_{xzxz} = \frac{1}{2}\frac{h - \sigma_{xx}}{u_{zz} - u_{xx}}.$$

$$u_{xz} \neq 0$$

$$C_{xyxy} = \frac{u_{zz} - u_{yy}}{u_{xx} - u_{yy}} C_{yzyz} = \frac{1}{2}\frac{h}{\Delta};$$

$$C_{xzxz} = (C_{zzzz} - 2C_{xxzz} + C_{xxxx})\frac{u_{xz}}{(u_{zz} - u_{xx})^2};$$

$$C_{xxxz} = (C_{xxzz} - C_{xxxx})\frac{u_{xz}}{u_{zz} - u_{xx}};$$

$$\Delta = (u_{xx} - u_{yy})(u_{zz} - u_{yy}) - u_{xz}^2$$

Ward Identities for C'_{ijkl}

$$u_{xz} = 0$$

$$C'_{xyxy} = K'_{xyxy} = \frac{1}{\det \underline{\Lambda}} \frac{\Lambda_{xx}^2 \Lambda_{yy}^2 \sigma_{xx}}{\Lambda_{xx}^2 - \Lambda_{yy}^2};$$

$$C'_{yzyz} = K'_{yzyz} = \frac{1}{\det \underline{\Lambda}} \frac{\Lambda_{yy}^2 \Lambda_{zz}^2 h}{\Lambda_{zz}^2 - \Lambda_{yy}^2};$$

$$C'_{xzxz} = K'_{xzxz} = \frac{1}{\det \underline{\Lambda}} \frac{\Lambda_{zz}^2 \Lambda_{xx}^2 (h - \sigma_{xx})}{\Lambda_{zz}^2 - \Lambda_{xx}^2}$$

When u_{xz} is nonzero, expressions are complicated

Minimal Model

$$f_{\text{iso}} = \tfrac{1}{2} B u_{\alpha\alpha}^2 + \frac{1}{2} r \mathrm{Tr}\underline{\tilde{u}}^2 - w \mathrm{Tr}\underline{\tilde{u}}^3 + v\left(\mathrm{Tr}\underline{\tilde{u}}^2\right)^2$$

Ye, et al. PRL **98**, 147801 (2007)

$$\tilde{u}_{\alpha\beta} = u_{\alpha\beta} - \tfrac{1}{3}\delta_{\alpha\beta} u_{\gamma\gamma}$$

$$B \to \infty \Rightarrow u_{\alpha\alpha} = 0$$

$f_{\text{iso}} \to$ Maier - Saupe - de Gennes

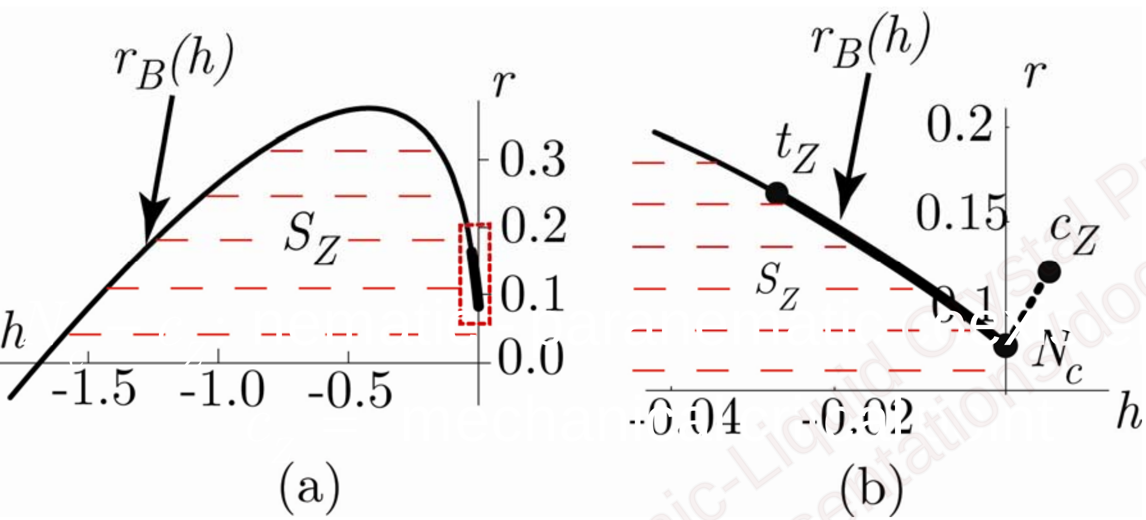
- Not full incompressibility
- Nematic in crossed electric and magnetic fields

$$g = f_{\text{iso}}(\underline{\tilde{u}}) - h u_{zz} - \sigma_{xx} u_{xx}$$

$$\sigma_{ij} = \frac{\partial f}{\partial u_{ij}} :$$ 2^{nd} Piola - Kirchhoff Stress tensor

Return to Eng. and Cauchy stress later

Phase diagram at $\sigma_{xx}=0$



$$\tilde{u} = \begin{pmatrix} -\frac{S}{3}+\eta_1 & \eta_2 & 0 \\ \eta_2 & -\frac{S}{3}-\eta_1 & 0 \\ 0 & 0 & \frac{2S}{3} \end{pmatrix}$$

$h > 0 : S > 0$

$h < 0 : S < 0$

S_Z : Biaxial coexistence - $\vec{\eta} = (\eta_1, \eta_2) = \eta(\cos 2\theta, \sin 2\theta)$

η attains equilibrium value

t_Z : Tricritical point; $N_Z - t_Z$-First - order line

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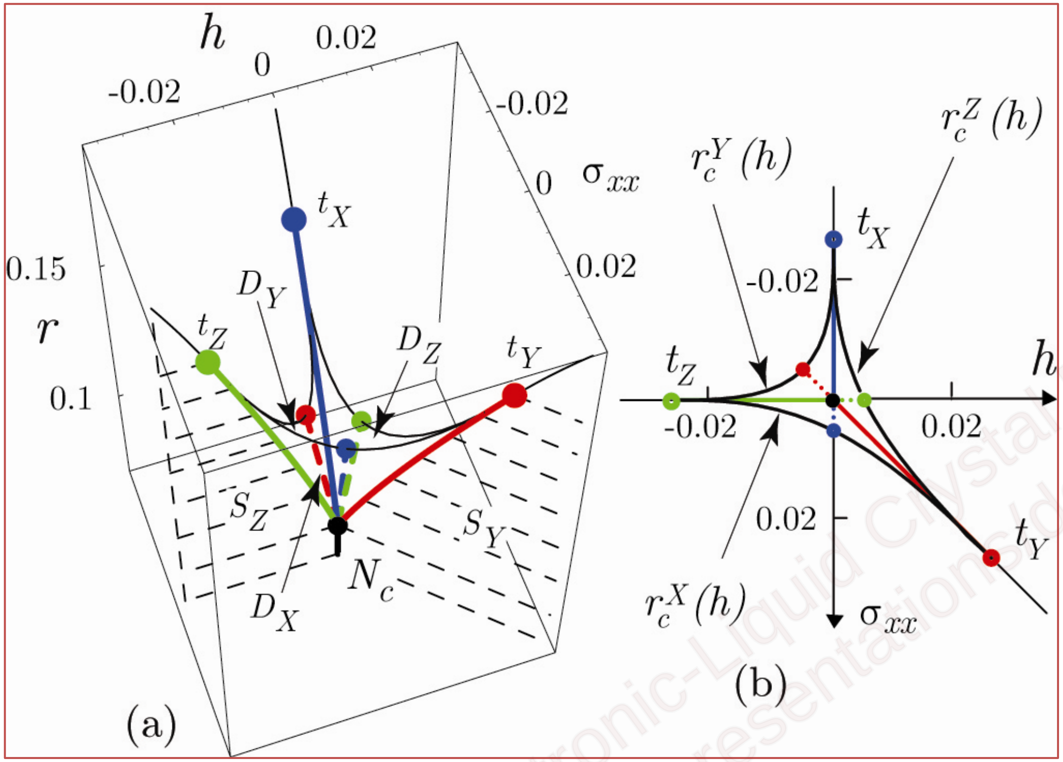
Global Phase Digram



D_X, D_Y, D_Z : Coexistence of discrete biaxial phases

S_X, S_Y, S_Z : Coexistence of continuous biaxial phases

$$\underline{\tilde{u}} = \begin{pmatrix} -\frac{S}{3}+\eta_1 & 0 & \eta_2 \\ 0 & \frac{2S}{3} & 0 \\ \eta_2 & 0 & -\frac{S}{3}-\eta_1 \end{pmatrix}$$

Order parameter near S_Y with $S < 0$

$$hu_{zz} + \sigma_{xx} u_{xx} = -hu_{yy} + (\sigma_{xx} - h) u_{xx}$$

$$g(r, h, \sigma_{xx}) = g(r, -h, \sigma_{xx} - h)$$

Plane $h = \sigma_{xx}$ $\equiv$ Planes $h = 0, \sigma_{xx} = 0$

R.G. Priest, Phys. Lett. A 47 475 (1974); Frisken, Bergersen, Palffy-Muhoray, Mol. Cryst. Liq. Cryst. 148, 45 (1987).

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Modulus at "Constant Stress"

$$xx = 1;\ yy = 2;\ zz = 3;$$
$$xz = 4;\ zy = 5;\ xy = 6.$$

$$\underline{K}' = \begin{pmatrix} \sigma'_{xx} + C'_{11} & C'_{13} & C'_{14} & 0 \\ C'_{13} & C'_{33} & C'_{34} & 0 \\ C'_{14} & C'_{34} & C'_{44} & 0 \\ 0 & 0 & 0 & C'_{55} \end{pmatrix}$$

$$\sigma_{yy} = 0$$
$$\delta\Lambda_{ij} = 0 \text{ for } ij \neq xz; yy \qquad K'^{1\sigma}_{xzxz} = C'_{44}$$
$$\delta\Lambda_{ij} = 0 \text{ for } ij \neq yz; yy \qquad K'^{1\sigma}_{yzyz} = C'_{55}$$

$$\sigma_{yy} = \delta\sigma_{zz} = \delta\sigma_{xx} = 0$$
$$\delta\Lambda_{ij} = 0 \text{ for } ij \neq xx, yy, zz, xz \qquad K'^{2\sigma}_{xzxz} = \frac{\det \underline{K}'_{||}}{C'_{33}(\sigma'_{xx} + C'_{11}) - (C'_{13})^2}$$
$$\delta\Lambda_{ij} = 0 \text{ for } ij \neq xx, yy, zz, yz \qquad K'^{2\sigma}_{yzyz} = C'_{55}$$

Moduli Again

The general behavior of Elastic moduli shown below is independent of model details.