

Spectral decomposition in anisotropic liquids

Valery S. Volkov

Institute of Petrochemical Synthesis, Russian Academy of Sciences,
Moscow 117912, Russia e-mail:
vsvolk@mail.ru

Abstract

The spectral approach to the Maxwell liquids with arbitrary anisotropy is presented. It is based on the spectral decomposition of viscosity and relaxation time tensors. In this way the general structure of anisotropic viscoelasticity becomes quite transparent. Special cases of viscous and relaxation spectra are considered. They reduce the number of independent viscoelastic parameters. The viscoelastic orientation dynamics is also discussed. The symmetry and thermodynamic analysis show that the nematic viscoelasticity of Maxwell type is characterized by the second-order director equation. This equation gives the correct results in viscous and elastic limits. The spectral theory of soft (zero-energy) viscoelastic eigenmodes is developed. They have predicted for complex anisotropic liquids. The present structure-continual theory of anisotropic liquids can be applied for analyzing flows of LC polymer systems and polymer suspensions with particles of arbitrary shape.

I. Introduction

The rheology of media with microstructure has a long history [1-13]. In addition to the classical translation degree, their material points possess internal degrees of freedom. Examples of non-classical continua include polymer systems, suspensions and emulsions, composite materials, liquid crystals, magnetic and physiological liquids, etc. For polymer media relaxation phenomena play a central role [14-16]. The appearance of polymeric liquid crystals inspires the investigation of the anisotropic viscoelastic liquids. As the rigorous microscopic theory for thermotropic LC polymers is still absent much attention has been paid to various macroscopic models [17-27]. Other important materials with anisotropic viscoelastic properties are polymer nanocomposites and LC elastomers [28-30]. The anisotropic character of viscoelastic properties is a feature of many directed polymer systems.

The effect of relaxation anisotropy was taken into account in the structure-continual theory for anisotropic viscoelastic liquids [18, 31]. Anisotropic relaxation is described by the characteristic relaxation time tensor. This theory combines microstructural information with continuum representation. The internal structure of liquids is specified by introducing the additional continual variables. Thermodynamics fundamental to viscoelastic liquids with uniaxial anisotropy was developed by Leonov and Volkov [32]. This approach leads to important conclusion that the relaxation anisotropy can be described in a natural way within the framework of classical non-equilibrium thermodynamics. It involves the introduction of a tensor internal parameter to characterize elastic deformation of the microstructure. The basic problem is to determine the general structure of anisotropic viscosity and relaxation time. They are the fourth-rank tensors.

Physical properties of media are characterized by mathematical quantities called tensors. A non-directional property, such as viscosity of incompressible Newtonian liquid, can be specified by a scalar, or zero rank tensor. Tensor properties of higher rank also exist. The fourth-rank tensors are commonly used in the anisotropic elasticity and computational mechanics. They were introduced for specifications of anisotropic elastic properties of crystals [33]. The mathematical treatment of such tensors is difficult. There are various decompositions of fourth-rank tensors into independent elementary tensors. In the simplest form the general structure of fourth-rank tensor is defined by the spectral decomposition into its eigentensors [34]. In many physical situations, this decomposition is of key importance. The spectral (algebraic) approach to the symmetric anisotropic elasticity was first presented by Thomson (Lord Kelvin) [35, 36]. For the generally anisotropic solid, Kelvin has introduced the six principal strains and principal elasticities. Pipkin [37] and Rychlewski [38, 39] reconsidered this approach in the context of

modern tensor algebra. At present, the spectral approach to the symmetric anisotropic elasticity has received increasing attention [40]. However, relatively little has been done for liquids. For incompressible uniaxial liquids, the principal viscosities and relaxation times were introduced by Volkov and Kulichikhin [18, 41, 42].

The aim of this paper is to develop the spectral theory of momentless Maxwell's liquids with arbitrary anisotropy. It is based on the spectral decomposition of the anisotropic viscosity and the anisotropic relaxation time. The important special cases of viscous and relaxation spectra are considered. We give a spectral interpretation of the soft viscoelastic modes. The present work is primarily concerned with constitutive equations. In the last section the viscoelastic orientation equation is considered. This theory allows expressing the basic ideas of anisotropic viscoelastic material behavior.

II. Spectral decomposition

For viscoelastic media, it is common to decompose the stress tensor into reversible and irreversible parts

$$\sigma_{ij} = \sigma_{ij}^r + \tilde{\sigma}_{ij} . \tag{1}$$

In general, liquids are defined as media with purely isotropic reversible stresses

$$\sigma_{ij}^r = -p\delta_{ij} . \tag{2}$$

The scalar p is called the pressure. Thus liquids at rest should only exert normal pressure across any plain. Under a rigid body motion (including rest) they cannot sustain any shearing stresses.

Consider the anisotropic viscoelastic liquids with *symmetric* stresses. The basic constitutive equation for these momentless liquids has the simple form [18]:

$$\tau_{ijke} \frac{D\tilde{\sigma}_{ke}}{Dt} + \tilde{\sigma}_{ij} = \eta_{ijke} e_{ke} , \tag{3}$$

where $e_{ij} = (v_{i,j} + v_{j,i})/2$ is the symmetric part of velocity gradient. For specification of anisotropic viscoelastic properties, the symmetric viscosity η_{ijke} and relaxation time τ_{ijke} tensors were introduced. This constitutive equation for anisotropic viscoelasticity is an extension of the Maxwell relation to anisotropic liquids.

Because the strain rate tensor e_{ij} and the dynamic stress tensor $\tilde{\sigma}_{ij}$ are symmetric, there is no loss of generality in taking η_{ijke} and τ_{ijke} to have internal symmetries

$$\eta_{ijke} = \eta_{jike} = \eta_{ijek}, \quad \tau_{ijke} = \tau_{jike} = \tau_{ijek}. \quad (4)$$

For simplification, we will assume that these tensors have the major symmetry

$$\eta_{ijke} = \eta_{keij}, \quad \tau_{ijke} = \tau_{keij}. \quad (5)$$

The Jaumann time derivative $D\sigma_{ij}/Dt$ is commonly used in continual theories for liquid crystals.

The constitutive equation (3) has a thermodynamic foundation [43]. This thermodynamic approach to the viscoelasticity is based on the natural generalization of the classical non-equilibrium thermodynamics developed by Onsager *et al.* [44] to the systems with memory [45]. Equation (3) is the simple case when the thermodynamic force e_{ij} and flux σ_{ij} are related by the anisotropic relaxation equation of the first order. The relaxation thermodynamics provides the investigation of viscoelasticity without using the particular stress – “strain” relation for liquids. The viscoelasticity can be naturally incorporated into a thermodynamic treatment.

The spectral analysis of tensors is a simplest mathematical tool to study the general structure of the anisotropic viscosity η_{ijke} and the anisotropic relaxation time τ_{ijke} . It is well-known that the most important things of a tensor are its eigenvalues and eigentensors. By Sylvester’s theorem, any symmetric fourth-rank tensor can be decomposed into eigentensors. This theorem provides a rapid way to calculate functions of a matrix associated with tensor. For the general viscosity and relaxation time tensors, when spectrally decomposed, the following relations hold

$$\eta_{ijke} = \sum_{\alpha=1}^n \eta_{\alpha} a_{ijke}^{\alpha}, \quad \tau_{ijke} = \sum_{\alpha=1}^n \tau_{\alpha} a_{ijke}^{\alpha}, \quad n \leq 6, \quad (6)$$

where n is the number of pairwise distinct eigenvalues η_{α} , τ_{α} . Calculating the decompositions (6) consists of finding the eigenvalues and eigentensors a_{ijke}^{α} for concrete liquids. The eigenvalues of any real symmetric tensor are real. In the general case of a extremely complex anisotropic liquid, we have $n = 6$. The symmetric tensors η_{ijke} and τ_{ijke} are physically realizable

if and only if their eigenvalues η_α, τ_α are nonnegative. These invariant viscoelastic constants are called *principal* viscosities and principal relaxation times. They may or may not be all distinct. It should be noted that the individual components of the viscosity and relaxation time tensors do not satisfy invariance conditions.

The idempotent and “perpendicular” basis tensors a_{ijke}^α satisfy the following equations

$$I_{ijke} = \sum_{\alpha=1}^n a_{ijke}^\alpha, \tag{7}$$

$$a_{ijke}^\alpha a_{kemn}^\beta = 0 \quad \text{for } \alpha \neq \beta,$$

$$a_{ijke}^\alpha a_{kemn}^\alpha = a_{ijmn}^\alpha.$$

The identity tensor I_{ijke} may be expanded in terms of the eigentensors a_{ijke}^α . Any power of eigentensor is the tensor itself. This property is called idempotence (Latin for self-power). There is the Lagrange-Sylvester formula which enables to compute eigentensors directly. If the principal viscosities η_α are distinct, then we have

$$a_{ijke}^\alpha = \prod_{\beta=1}^6 \frac{\eta_{ijke} - \eta_\beta I_{ijke}}{\eta_\alpha - \eta_\beta}, \quad \beta \neq \alpha \tag{8}$$

In this case the eigentensors are well determined.

According to the spectral decomposition (6, 7), the general symmetric viscoelasticity of Maxwell type is described by six principal viscosity coefficients η_α and six principal relaxation times τ_α . They represent the point spectra of the anisotropic viscosity and relaxation time. If two or more eigenvalues of material tensor are identical, then the spectrum is called degenerate. The concept of spectral decomposition extremely simplifies an analysis of anisotropic viscoelasticity. It facilitates the computation of sums, products, and inverses for the symmetric fourth-rank material tensors. To compute the inverse, we simply take the reciprocal of eigenvalues. If τ_{ijke} has no zero eigenvalues, then

$$\tau_{ijke}^{-1} = \sum_{\alpha=1}^n \frac{1}{\tau_\alpha} a_{ijke}^\alpha. \tag{9}$$

Anisotropic viscous (non-polymer) liquids can be considered as the limit case of viscoelastic liquids (3). Their relaxation times τ_α relative to the time scale of flow are short. Therefore, these relaxation processes are negligible. The stress of anisotropic viscous liquids is defined by the generalized Newton law

$$\tilde{\sigma}_{ij} = \eta_{ijkn} e_{kn}. \quad (10)$$

This equation simply means that each stress component is linearly related to all components of strain rate tensor. Such behavior is typical for suspensions of anisotropic particles in viscous liquids [46]. An example which requires 5 viscosity coefficients is a dilute suspension of rigid ellipsoids. If the anisotropy has an axial symmetry, then 3 independent viscosity coefficients will suffice in incompressible case. Anisotropic solids can be considered as the inverse limit case. They are of media with extremely long relaxation times. In this case, the stress is determined by the generalized Hook law

$$\sigma_{ij} = G_{ijkn} \varepsilon_{kn}, \quad (11)$$

where G_{ijke} is the elasticity tensor and ε_{ij} is the symmetric strain tensor. Solids of this type are crystals and composite materials. The physically most important cases lie between these two extremes.

It is useful to introduce the “principal” stresses σ_{ij}^α and the “principal” strain rates e_{ij}^α . They are defined as follows

$$\sigma_{ij}^\alpha = a_{ijke}^\alpha \tilde{\sigma}_{ke}, \quad e_{ij}^\alpha = a_{ijke}^\alpha e_{ke}. \quad (12)$$

Using Eq. (7) and relations $I_{ijke} \sigma_{ke} = \sigma_{ij}$, $I_{ijke} e_{ke} = e_{ij}$, it is easily to obtain the following decompositions

$$\tilde{\sigma}_{ij} = \sum_{\alpha=1}^n \sigma_{ij}^\alpha, \quad e_{ij} = \sum_{\alpha=1}^n e_{ij}^\alpha, \quad n \leq 6. \quad (13)$$

The “principal” stresses and strain rates are mutually perpendicular

$$\sigma_{ij}^\alpha \sigma_{ij}^\beta = 0, \quad e_{ij}^\alpha e_{ij}^\beta = 0 \quad \text{for } \alpha \neq \beta. \quad (14)$$

The basic law of anisotropic viscoelasticity (3) can be expressed as orthogonal decomposition

$$\begin{aligned} \tilde{\sigma}_{ij} &= \sum_{\alpha=1}^n \sigma_{ij}^{\alpha}, & (15) \\ \tau_{\alpha} \frac{D\sigma_{ij}^{\alpha}}{Dt} + \sigma_{ij}^{\alpha} &= \eta_{\alpha} e_{ij}^{\alpha}. \end{aligned}$$

These equations define the relaxation eigenmodes for anisotropic Maxwell liquids. They are sufficiently general to include liquids of any symmetry. For the general anisotropy, we have six coupled Maxwell modes. In the linear region, these relaxation modes are separated.

It is important that the viscous and relaxation spectra η_{α} , τ_{α} , $\alpha \leq 6$ can be experimentally observed. There are restrictions on the above eigen-material constants due to thermodynamic arguments

$$\begin{aligned} \eta_1 \geq 0, \dots, \eta_n \geq 0, & & (16) \\ \tau_1 \geq 0, \dots, \tau_n \geq 0, & \quad n \leq 6. \end{aligned}$$

These conditions have the simple form. The zero eigenvalues define the ideal anisotropic liquids with different *soft modes*. In this case, some viscoelastic eigenmodes make no contribution to the stress. The ideal isotropic liquids are characterized by the trivial condition: $\eta_{\alpha} = 0$ and $\tau_{\alpha} = 0$ for $\alpha \leq 6$. These simplest liquids are nondissipative in all material directions. For non-viscous liquids and perfect gases, such behavior is typical. The liquids with soft modes are able to flow without resistance.

III. Special cases of viscous spectrum

The purpose of this section is to discuss the application of general approach to some simple cases. It is important to define the structure of eigentensors in the main spectral formulae (6) for different liquids. The simplest example is a *compressible* Newtonian liquid. For this liquid, the spectral decomposition of viscosity tensor has the simple form

$$\eta_{ijke} = \eta_1 a_{ijke}^1 + \eta_2 a_{ijke}^2, \quad (17)$$

The isotropic eigentensors are expressed by

$$a_{ijke}^1 = I_{ijke} - \frac{1}{3} \delta_{ij} \delta_{ke}, \quad a_{ijke}^2 = \frac{1}{3} \delta_{ij} \delta_{ke},$$

where $I_{ijke} = (\delta_{ik} \delta_{je} + \delta_{ie} \delta_{jk}) / 2$ is the symmetric unit tensor. Newtonian's liquid is characterized by two principal viscosities η_1 and η_2 . Using Eq. (17), we obtain the constitutive equation

$$\sigma_{ij} + p \delta_{ij} = 2\eta \left(e_{ij} - \frac{1}{3} e_{mn} \delta_{ij} \right) + \zeta e_{mn} \delta_{ij}, \quad (18)$$

where $\eta = \eta_1 / 2$ is the shear viscosity, $\zeta = \eta_2 / 3$ is the second (bulk) viscosity, and e_{mn} is the volumetric strain rate. Equation (18) completely describes a compressible Newtonian liquid [47]. This linearly viscous liquid has no directional preference. For many applications, the Stokes relation $\zeta = 0$ is sufficiently accurate. The historical aspects of this relation were discussed by Truesdell [48]. The liquid with $\zeta = 0$ is called the Stokes liquid. It causes no viscous resistance to volume flows. In some cases, the bulk viscosity may greatly exceed the shear viscosity [47]. For polymer liquids, the bulk viscosity is normally irrelevant.

The extremely important case of macroscopic anisotropy is the transverse isotropy. For *incompressible* liquids with uniaxial anisotropy, the spectral decomposition of the viscosity tensor has the form

$$\eta_{ijke} = \sum_{\alpha=1}^3 \eta_{\alpha} a_{ijke}^{\alpha}. \quad (19)$$

This relation is more complicated to discuss than (17). The anisotropic viscosity of these nematics is described by the three principal viscosities η_{α} . The nematic eigentensors are defined as

$$a_{ijke}^1 = I_{(ij)(ke)} - a_{ijke}^2 - a_{ijke}^3, \quad a_{ijke}^2 = \frac{3}{2} (n_{ij} - \frac{1}{3} \delta_{ij}) (n_{ke} - \frac{1}{3} \delta_{ke}),$$

$$a_{ijke}^3 = \frac{1}{2} (\delta_{ik}^{\perp} n_{je} + n_{ik} \delta_{je}^{\perp} + \delta_{ie}^{\perp} n_{jk} + n_{ie} \delta_{jk}^{\perp}). \quad (20)$$

Here \mathbf{n} denotes the director and $I_{(ij)(ke)} = (\delta_{ik} \delta_{je} + \delta_{ie} \delta_{jk}) / 2 - \delta_{ij} \delta_{ke} / 3$ is the traceless unit tensor. The tensor $n_{ij} = n_i n_j$ is called the structural tensor since it lay down the structural

symmetry. The transverse Kronecker symbol $\delta_{ij}^\perp = \delta_{ij} - n_i n_j$ makes a projection on the direction orthogonal to the unit vector \mathbf{n} . The first eigentensor can be written as follows

$$a_{ijke}^1 = \frac{1}{2} \left(\delta_{ik}^\perp \delta_{je}^\perp + \delta_{jk}^\perp \delta_{ie}^\perp - \delta_{ij}^\perp \delta_{ke}^\perp \right). \quad (21)$$

For incompressible uniaxial liquids, the spectral stress representation takes the form

$$\tilde{\sigma}_{ij} = \sum_{\alpha=1}^3 \sigma_{ij}^\alpha, \quad \sigma_{ij}^\alpha = \eta_\alpha e_{ij}^\alpha(\mathbf{n}). \quad (22)$$

The ‘‘principal’’ strain rates e_{ij}^α are defined by

$$\begin{aligned} e_{ij}^1 &= e_{ij} - e_{ij}^2 - e_{ij}^3, & e_{ij}^2 &= \frac{3}{2} n_{ke} e_{ke} \left(n_{ij} - \frac{1}{3} \delta_{ij} \right), \\ e_{ij}^3 &= n_{ik} e_{kj} + e_{ik} n_{kj} - 2n_{ijks} e_{ks}. \end{aligned} \quad (23)$$

The transversely isotropic liquids with soft modes are possible. They are defined by the zero viscosities in some principal directions of anisotropy. We note that the zero principal viscosities η_2 and η_3 define the two dissipative soft modes considered by Leonov and Volkov [49]. In this paper, the prediction of dissipative soft modes is based on a marginal stability analysis. These modes cause no resistance to flow, nullifying the corresponding components of the dissipative part of the stress tensor. It is useful to consider the viscous spectrum with a single principal viscosity: $\eta = \eta_1$, $\eta_2 = \eta_3 = 0$. This reduction in the number of macroscopic parameters is justified if values of η_2 and η_3 relative to η_1 are small. For such class of incompressible nematic liquids, the constitutive equation has the form

$$\tilde{\sigma}_{ij} = \eta \left[e_{ij} - \left(n_{ik} e_{kj} + e_{ik} n_{kj} \right) + \frac{1}{2} \left(n_{ij} + \delta_{ij} \right) n_{kn} e_{kn} \right]. \quad (24)$$

This equation simplifies the general flow analysis. In the non-symmetric viscous nematodynamics the problem of reduction in the number of Leslie’s viscosity coefficients was considered by McIntosh and Leslie [50].

Equation (22) can be reduced to the Ericksen constitutive equation [8]:

$$\sigma_{ij} + p\delta_{ij} = \mu_1 e_{ij} + \mu_2 I_0 n_{ij} + \mu_3 n_{k(i} e_{j)k}, \quad (25)$$

where μ_α are the viscosity constants and $I_0 = n_{ek} e_{ke}$ is the simultaneous invariant of tensors n_{ij} and e_{ij} . The condition of positive dissipation imposes additional constraints on the viscosity coefficients

$$\mu_1 \geq 0, \quad \mu_1 + 2(\mu_2 + \mu_3)/3 \geq 0, \quad \mu_1 + \mu_3/2 \geq 0. \quad (26)$$

There are linear relations between the principal viscosities η_α and Ericksen's (non-principal) viscosity coefficients μ_α :

$$\eta_1 = \mu_1, \quad \eta_2 = \mu_1 + 2(\mu_2 + \mu_3)/3, \quad \eta_3 = \mu_1 + \mu_3/2. \quad (27)$$

IV. Viscoelastic nematics

Let us consider the simplification that is direct consequence of the uniaxial anisotropy for viscoelastic liquids. For incompressible Maxwell's liquids with uniaxial anisotropy, the symmetric viscosity and relaxation time tensors are decomposed into three spectral components

$$\eta_{ijke} = \sum_{\alpha=1}^3 \eta_\alpha a_{ijke}^\alpha, \quad \tau_{ijke} = \sum_{\alpha=1}^3 \tau_\alpha a_{ijke}^\alpha \quad (28)$$

The eigenbasis a_{ijke}^α , $\alpha = 1, 2, 3$ are defined by relations (20). In the incompressible case, the uniaxial viscoelasticity of Maxwell type are described by three principal viscosities η_α and three principal relaxation times τ_α . They represent spectra of viscosity and relaxation time with uniaxial anisotropy. These linear spectra show the relative contribution of different components in the total anisotropic viscosity and anisotropic relaxation time. The spectral decomposition leads to a representation of the dynamic stress tensor as a sum of modal contributions of Maxwell type

$$\tilde{\sigma}_{ij} = \sum_{\alpha=1}^3 \sigma_{ij}^\alpha, \quad (29)$$

$$\tau_\alpha \frac{D\sigma_{ij}^\alpha}{Dt} + \sigma_{ij}^\alpha = \eta_\alpha e_{ij}^\alpha(\mathbf{n}).$$

The “principal” stresses σ_{ij}^α are mutually orthogonal. They are defined by

$$\begin{aligned}\sigma_{ij}^1 &= \sigma_{ij} - \sigma_{ij}^2 - \sigma_{ij}^3, & \sigma_{ij}^2 &= \frac{3}{2} n_{ke} \sigma_{ke} (n_{ij} - \frac{1}{3} \delta_{ij}), \\ \sigma_{ij}^3 &= n_{ie} \gamma_{ej} + \gamma_{ie} n_{ej} - 2n_{ijke} \sigma_{ke}.\end{aligned}\tag{30}$$

The “principal” strain rates e_{ij}^α are defined by the analogous formulae (23). Equations (29) define the relaxation eigenmodes which do not contain compressions. The uniaxial anisotropy leads to three coupled relaxation eigenmodes. In the linear region, they are independent. It is important that the slow and fast relaxation eigenmodes can be found in different principal material directions.

The structural formulae (28) allow selecting the important special cases. They lead to the reduction in the number of macroscopic parameters. The invariant viscoelastic constants η_α and τ_α depend on the nature of liquid. In the case of the isotropic Maxwell liquid, three principal viscosities and relaxation times are equal: $\eta_1 = \eta_2 = \eta_3$ and $\tau_1 = \tau_2 = \tau_3$. This is the case of triple coalescence of the eigenvalues of viscosity and relaxation time tensors. In a completely different context, the special uniaxial viscoelastic liquid was considered already by Volkov and Kulichikhin [18]. In this case we have the double coalescence of principal viscosities and relaxation times:

$$\eta_1 = \eta_2 \neq \eta_3, \quad \tau_1 = \tau_2 \neq \tau_3.\tag{31}$$

The viscoelastic properties of this uniaxial liquid are described by two perpendicular eigenmodes:

$$\tilde{\sigma}_{ij} = \sigma_{ij}^\parallel + \sigma_{ij}^\perp,\tag{32}$$

where σ_{ij}^\parallel , σ_{ij}^\perp are the longitudinal and transverse (with respect to the director) stresses. Each mode obeys

$$\begin{aligned}\tau_\parallel \frac{D_\parallel}{Dt} (\sigma_{ij}^\parallel + \sigma_{ij}^\perp) + \sigma_{ij}^\parallel &= \eta_\parallel e_{ij}^\parallel, \\ \tau_\perp \frac{D_\perp}{Dt} (\sigma_{ij}^\parallel + \sigma_{ij}^\perp) + \sigma_{ij}^\perp &= \eta_\perp e_{ij}^\perp,\end{aligned}\tag{33}$$

were $\eta_{\parallel}, \eta_{\perp}$ and $\tau_{\parallel}, \tau_{\perp}$ are the viscosities and relaxation times of the longitudinal and transverse relaxation eigenmodes, respectively. Here the following notations are introduced:

$$\eta_1 = \eta_2 = \eta_{\perp}, \quad \tau_1 = \tau_2 = \tau_{\perp}, \quad \eta_3 = \eta_{\parallel}, \quad \tau_3 = \tau_{\parallel}. \quad (34)$$

The longitudinal e_{ij}^{\parallel} and transverse e_{ij}^{\perp} strain rates are expressed as

$$e_{ij}^{\perp} = e_{ij} - e_{ij}^{\parallel}, \quad (35)$$

$$e_{ij}^{\parallel} = n_{ie} e_{ej} + e_{ie} n_{ej} - 2n_{ijke} e_{ke}.$$

The coherence of eigenmodes is caused by the Jaumann (nonlinear) derivative. The longitudinal D_{\parallel} / Dt and transverse D_{\perp} / Dt components of this derivative are defined by

$$\frac{D_{\perp} \sigma_{ij}}{Dt} = \frac{D \sigma_{ij}}{Dt} - \frac{D_{\parallel} \sigma_{ij}}{Dt}, \quad (36)$$

$$\frac{D_{\parallel} \sigma_{ij}}{Dt} = n_{ie} \frac{D \sigma_{ej}}{Dt} + \frac{D \sigma_{ie}}{Dt} n_{ej} - 2n_{ijke} \frac{D \sigma_{ke}}{Dt}.$$

In general, there is a coupling of longitudinal and transverse relaxations. The linear viscoelasticity of these viscoelastic nematics is described by two non-interacting relaxation eigenmodes

$$\tilde{\sigma}_{ij} = \sigma_{ij}^{\parallel} + \sigma_{ij}^{\perp},$$

$$\tau_{\parallel} \frac{d}{dt} \sigma_{ij}^{\parallel} + \sigma_{ij}^{\parallel} = \eta_{\parallel} e_{ij}^{\parallel}, \quad (37)$$

$$\tau_{\perp} \frac{d}{dt} \sigma_{ij}^{\perp} + \sigma_{ij}^{\perp} = \eta_{\perp} e_{ij}^{\perp}.$$

It follows from the above considerations that the spectral approach provides a systematic way for obtaining the constitutive equations for anisotropic liquids with different symmetry.

V. Orientation dynamics

The constitutive laws described above were derived for liquids with arbitrary orientation of anisotropy. In general, the principal directions of anisotropy in liquid may be altered by the flow. Therefore, the evolution equations for these basic directions must be considered.

To complete the theory of transversely isotropic liquid (25), Ericksen [8] *postulates* the first-order differential equation for director

$$\frac{Dn_i}{Dt} = \lambda_v (e_{is} n_s - n_{imn} e_{mn}), \quad (38)$$

where λ_v is the ‘tumbling’ parameter. The director equation (38) is the simplest type of equations which might be used to determine the orientation of anisotropic liquids. This director equation reflects the cross effect between the flow and the “internal” rotation (orientation). It is very reasonable for viscous liquids with transverse isotropy.

The Ericksen orientation equation (38) can be derived in the framework of the structure-continual theory for non-symmetric anisotropic liquids [42, 51]. This theory does not introduce the additional assumption of type (38). In the case of viscoelastic liquids with transverse isotropy, it leads to the orientation equation of the second order in time

$$\frac{Dn_i}{Dt} - \lambda_v \beta_{ijk} e_{jk} = \tau_{ij} \frac{DF_j^e}{Dt}, \quad (39)$$

$$F_i^e = \frac{Dn_i}{Dt} - \lambda_v \beta_{ijk} e_{jk}.$$

This equation reflects the effect of *relaxation interaction* [42] and coupling between orientation and flow. Here $\tau_{ij} = \tau_r \delta_{ij}^\perp$ and τ_r is the rotational relaxation time. The third-rank tensor β_{ijk} is symmetric in the exchange of its last two indices. The explicit expression of β_{ijk} is

$$\beta_{ijk} = (\delta_{ij}^\perp n_k + \delta_{ik}^\perp n_j). \quad (40)$$

The director equation for nematic viscoelastic liquid of Maxwell type (39) contains the viscous λ_v and elastic λ_e “tumbling” parameters. It is consistent with the thermodynamic formulation by Leonov and Volkov [32]. The viscoelastic orientation equation (39) has simple meaning. In the viscous limit, this equation reduces to the well-known Ericksen equation (38). In the elastic limit, it gives similar equation

$$\frac{Dn_i}{Dt} = \lambda_e (e_{is}n_s - n_{imn}e_{mn}). \quad (41)$$

It is the differential form of the orientation equation for nematic solids with internal rotation [52]. In general, the orientation dynamics of polymer (viscoelastic) nematics depends on the acceleration of the director. This effect can be neglected in the slow director dynamics. In this case, one can use the Ericksen orientation equation for viscoelastic case [18].

The microstructure of liquids presented by director \mathbf{n} is symmetric with respect the reflection in planes parallel and perpendicular to \mathbf{n} . For liquids with complex internal structures, it is necessary to introduce the tensor representation for orientation. The types of alignment can be classified according to the eigenvalues of the order parameter tensor S_{ij} . Due to its symmetry, it can be spectrally decomposed as

$$S_{ij} = S_1 n_i n_j + S_2 m_i m_j + S_3 l_i l_j, \quad (42)$$

$$\delta_{ij} = n_i n_j + m_i m_j + l_i l_j,$$

where \mathbf{n} , \mathbf{m} , \mathbf{l} are orthonormal eigenvectors of \mathbf{S} with corresponding eigenvalues S_α , $\alpha=1,2,3$. The eigenvalues S_α are scalar order parameters. They give a measure of different types of order. The extensibility of microstructure is described by the first invariant $I_1 = S_{ii}$. In the case of uniaxial orientation state two eigenvalues coincide. The most general form of uniaxial order parameter is

$$S_{ij} = S_\perp (\delta_{ij} - n_i n_j) + S_\parallel n_i n_j, \quad (43)$$

where S_{\parallel} and S_{\perp} are the longitudinal and transverse order parameters. For polymer media with completely rigid macromolecules or anisotropic particles, the order parameter tensor is defined by

$$S_{ij} = S(n_i n_j - \frac{1}{3} \delta_{ij}), \quad (44)$$

where $S = S_{\parallel} - S_{\perp}$ is the well-known scalar order parameter. In this particular case, the order parameter tensor is traceless [13].

VI. Conclusions

In this paper, the spectral concept has introduced to represent the anisotropic Maxwell liquids with general anisotropy. Mathematical reasons enable to establish the general structure of the anisotropic viscosity and the anisotropic relaxation time. It was shown that the necessary parameters invariantly describing the rheological properties of anisotropic viscoelastic liquids may be derived by the spectral decomposition of the viscosity and relaxation time tensors. This approach extremely simplifies an analysis of anisotropic viscoelasticity. It is sufficiently general to include liquids of any symmetry.

The important particular cases of anisotropy were considered too. They lead to the reduction in the number of macroscopic parameters. The symmetry and thermodynamic analysis show that the nematic viscoelasticity of Maxwell type is characterized by the director equation of order greater than one.

The spectral theory of anisotropic viscoelastic liquids leads to the modal analysis. The magnitude of relaxation properties depends on direction in a liquid. Therefore the relaxation must be defined in relation to a direction in a liquid, and the magnitude of the relaxation may be different in different directions. For the anisotropic polymer liquids, the interaction of slow and fast relaxation processes in different material directions is fundamental. The soft viscoelastic eigenmodes have predicted for complex anisotropic liquids. They also significantly decrease the number of material parameters.

The eigenmode coefficients are called the principal viscosities and the principal relaxation times. They can be introduced for liquids with any anisotropy. The eigen-material constants are nonnegative. In general, it is possible to represent the anisotropic viscosity by six principal viscosity coefficients. For the important case of incompressible liquid with uniaxial anisotropy, three principal viscosities will suffice. These principal viscosities and Ericksen (non-principal) viscosities are related by linear relations.

We finally express that the present theory of anisotropic viscoelasticity is structure-continual. It is possible to consider the problem from a microscopic starting point. The phenomenological and microscopic approaches complement each other. The first approach is more general. It provides the theoretical basis for fundamental (tensor) experiments. The structural theories are based on a specifying modeling of media. They make possible the calculation of macroscopic parameters. For anisotropic polymer liquids, the microscopic approach is growing in importance. This is due to the fact that they are tensor viscoelastic properties. Therefore the viscoelasticity of anisotropic polymer liquids are characterized by essentially higher number of macroscopic parameters as compared with isotropic polymer systems. The additional parameters describe the effect that the viscoelastic properties vary with direction within a liquid. Unfortunately, for complex media frequently the phenomenological approach is the only possible.

The complication of structure leads to a prompt loss of symmetry. Spectral theory of anisotropic viscoelasticity opens possibility of comparing anisotropic liquids not connected with their symmetry at all. It offers a framework for extended and more quantitative classification of anisotropic liquids. The main structural formulae (6) reduces this problem to the construction of reasonable classifications for viscous and relaxation spectra.

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